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Total outer-connected domination numbers of trees

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ABSTRACT

Let $G = (V, E)$ be a graph without an isolated vertex. A set $D \subseteq V(G)$ is a *total dominating set* if D is dominating, and the induced subgraph $G[D]$ does not contain an isolated vertex. The total domination number of G is the minimum cardinality of a total dominating set of G. A set $D \subseteq V(G)$ is a *total outer-connected dominating set* if D is total dominating, and the induced subgraph $G[V(G) - D]$ is a connected graph. The total outer-connected domination number of G is the minimum cardinality of a total outer-connected dominating set of G. We characterize trees with equal total domination and total outer-connected domination numbers. We give a lower bound for the total outer-connected domination number of trees and we characterize the extremal trees.

1. Introduction

Let $G = (V, E)$ be a simple graph with $|V(G)| = n(G)$ and $|E(G)| = m(G)$. The *neighbourhood* $N_G(v)$ of a vertex v is the set of all vertices adjacent to v in G. The *degree* $d_G(v)$ of a vertex v is the number of edges incident with v in G, $d_G(v) = |N_G(v)|$. If $d_G(v) = 0$, then we call v an *isolate vertex*. Let $\Omega(G)$ be the set of all leaves of G, that is the set of vertices of degree 1, and let $n_1(G)$ be the cardinality of $\Omega(G)$. A vertex v is called a *support vertex* if v is a neighbour of a leaf and $d_G(v) > 1$. Denote by $S(G)$ the set of all support vertices in G and let $n_S(G)$ be the cardinality of $S(G)$. For notational convenience we denote $\Omega(G) \cup S(G)$ by $J(G)$. The *diameter* $\mathrm{diam}(G)$ of a connected graph G is the maximum distance between two vertices of G, that is $\mathrm{diam}(G) = \max_{u,v \in V(G)} d_G(u, v)$.

A set $D \subseteq V(G)$ is a *dominating set (DS) of* G if for every vertex $v \in V(G) - D$, there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a dominating set in G is the *domination number* denoted $\gamma(G)$. A minimum DS of a graph G is called a $\gamma(G)$-set.

If G is without an isolated vertex, then a set $D \subseteq V(G)$ is a *total dominating set (TDS) of* G if for every vertex $v \in V(G)$, there exists a vertex $u \in D$ such that v and u are adjacent. The minimum cardinality of a total dominating set in G is the *total domination number* denoted $\gamma_t(G)$. A minimum TDS of a graph G is called a $\gamma_t(G)$-set.

If G is without an isolated vertex, then a set $D \subseteq V(G)$ is a *total outer-connected dominating set (TCDS) of* G if D is total dominating set of G and the subgraph induced by $V(G) - D$ is connected. The minimum cardinality of a total outer-connected dominating set in G is the *total outer-connected domination number* denoted $\gamma_{tc}(G)$. A minimum TCDS of a graph G is called a $\gamma_{tc}(G)$-set. The total outer-connected domination number of a graph is first defined in [1]. For an application, we consider a computer network in which a core group of fileservers has the ability to communicate directly with every computer outside the core group. In addition, each fileserver is directly linked to at least one other fileserver and every two computers outside the core group may communicate with each other without intervention of any of the fileservers to protect the fileservers from overloading. A smallest core group with these properties is a γ_{tc}-set for the graph representing the network.

A set $S \subseteq V$ is a 2-*packing* if for each pair of vertices $u, v \in S$, $N_G[u] \cap N_G[v] = \emptyset$.

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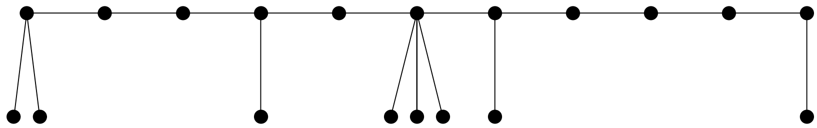

Fig. 1. Caterpillar.

We denote a path on n vertices by $P_n = (v_1, \ldots, v_n)$. A *caterpillar* is a tree with the property that the removal of its leaves results in a path, called the *spine* of the caterpillar (Fig. 1).

For any graph theoretical parameters λ and μ, we define G to be (λ, μ)-graph if $\lambda(G) = \mu(G)$. In this paper we provide a constructive characterization of (γ_t, γ_{tc})-trees. We give a lower bound for the total outer-connected domination numbers of trees and we characterize the extremal trees. For any unexplained terms and symbols see [2].

2. A characterization of (γ_t, γ_{tc})-trees

In our characterization of (γ_t, γ_{tc})-trees we will need the following two observations. The first one has been taken from [3].

Observation 1 ([3]). *Let T be a tree that is not a star. Then there exists a $\gamma_t(T)$-set that contains no leaf.* ■

Observation 2. *Let T be a tree with $n(T) \geq 3$. Then each support vertex is in every $\gamma_t(T)$-set.* ■

Before we give and prove the main result of this section, we make the following observation for a graph G and the total outer-connected domination number.

Proposition 3. *If G is a graph without an isolated vertex and $\gamma_{tc}(G) \leq n(G) - 2$, then each leaf and each support vertex belong to every minimum total outer-connected dominating set of G.*

Proof. Since every total outer-connected dominating set is a total dominating set and every support vertex belongs to every total dominating set, we conclude that each support vertex belong to every total outer-connected dominating set. Therefore, if a leaf does not belong to a minimum total outer-connected dominating set D of G, then since $G[V(G) - D]$ is connected, $\gamma_{tc}(G) = |D| = n(G) - 1$, a contradiction. □

Let $\mathcal{O}$ be the following operation defined on a tree T.

- **Operation** $\mathcal{O}$. Assume $x \in V(T) - J(T)$. Then add a path (y_1, y_2, y_3) and the edge xy_1.

Let $\mathcal{T}$ be the family of trees such that $\mathcal{T} = \{T : T$ is obtained from P_6 by a finite sequence of operations $\mathcal{O}\} \cup \{P_2, P_3\}$ (Fig. 2).

We show first that each tree in the family $\mathcal{T}$ has equal total domination number and total outer-connected domination number. We begin with a straightforward observation.

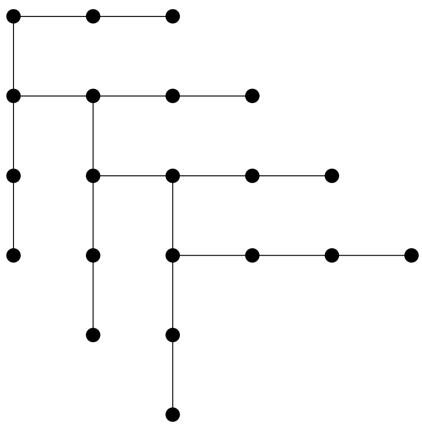

Fig. 2. Tree T belonging to the family $\mathcal{T}$.

Observation 4. *If a tree T with $n(T) \geq 4$ belongs to the family $\mathcal{T}$, then*

1. *Each vertex of $S(T)$ is of degree 2.*
2. *$S(T)$ is a dominating set of T.*
3. *If u and v are distinct vertices of $S(T)$, then $d_T(u, v) \geq 3$ and therefore $S(T)$ is a 2-packing.*

Corollary 5. *If T with $n(T) \geq 4$ belongs to the family $\mathcal{T}$, then $\gamma(T) = |S(T)|$ and $\gamma_t(T) = |J(T)|$.*

Lemma 6. *If a tree T belongs to the family $\mathcal{T}$, then T is a (γ_t, γ_{tc})-tree.*

Proof. We proceed by induction on the number of operations $s(T)$ required to construct the tree T. If $s(T) = 0$, then $T \in \{P_2, P_3, P_6\}$ and obviously T is a (γ_t, γ_{tc})-tree. Moreover, if $T = P_6$, then $J(T)$ is a $\gamma_{tc}(T)$-set. Assume now that T is a tree belonging to the family $\mathcal{T}$ with $s(T) = k$ for some positive integer k and each tree $T' \in \mathcal{T}$ with $s(T') < k$ and with $n(T') \geq 4$ is a (γ_t, γ_{tc})-tree in which $J(T')$ is a $\gamma_{tc}(T')$-set. Then T can be obtained from a tree T' belonging to $\mathcal{T}$ by operation $\mathcal{O}$, where $x \in V(T') - J(T')$ and we add a path (y_1, y_2, y_3) and the edge xy_1. Then y_3 is a leaf in T and y_2 is a support vertex and thus $J(T) = J(T') \cup \{y_2, y_3\}$. Moreover, $V(T) - \{x, y_1\}$ is a total outer-connected dominating set of T and for this reason Proposition 3 implies that y_2 and y_3 belong to every $\gamma_{tc}(T)$-set. Hence $\gamma_{tc}(T) \geq |J(T)| = |J(T')| + 2 = \gamma_{tc}(T') + 2$. Since x does not belong to a $\gamma_{tc}(T')$-set, namely $J(T')$, we conclude that $\gamma_{tc}(T) = \gamma_{tc}(T') + 2$. By the induction hypothesis and Corollary 5, $\gamma_{tc}(T') = \gamma_t(T') = |J(T')|$. In this way $\gamma_{tc}(T) = |J(T)|$ and in particular, $\gamma_t(T) = \gamma_{tc}(T)$. □

We show next that every (γ_t, γ_{tc})-tree belongs to the family $\mathcal{T}$.

Lemma 7. *If T is a (γ_t, γ_{tc})-tree, then T belongs to the family $\mathcal{T}$.*

Proof. It is easy to verify that the statement is true for all trees T with diameter less than 5. For this reason from now on we consider only trees T with diameter greater than or equal to 5.

Let T be a (γ_t, γ_{tc})-tree. We proceed by induction on the number of vertices $n(T)$ of a (γ_t, γ_{tc})-tree. Let T be a (γ_t, γ_{tc})-tree and assume that the result holds for all trees on $n(T) - 1$ and fewer vertices. Let $P = (s_0, s_1, \ldots, s_l)$, $l \geq 5$, be a longest path in T and let D_{tc} be a $\gamma_{tc}(T)$-set. As T is a (γ_t, γ_{tc})-tree and $n(T) \geq 6$, we conclude that $\gamma_{tc}(T) \leq n(T) - 2$ and thus by Proposition 3 all leaves and all support vertices of T belong to D_{tc}. If $d_T(s_1) > 2$, then s_1 is a neighbour of at least two leaves of T. Then $D_{tc} - \{s_0\}$ is a TDS of T and so $\gamma_{tc}(T) > \gamma_t(T)$, a contradiction. Hence $d_T(s_1) = 2$. Similarly, if s_2 is a support vertex, then again $D_{tc} - \{s_0\}$ is a TDS of T, contradiction.

Suppose $d_T(s_2) > 2$ and let A be the set of support vertices adjacent to s_2 and let B be the set of all leaves which are neighbours of any vertex belonging to A. Then $A \cup B \subseteq D_{tc}$, but $(D_{tc} - B) \cup \{s_2\}$ is a TDS of T of cardinality smaller than $\gamma_t(T)$, a contradiction. Therefore $d_T(s_1) = d_T(s_2) = 2$.

Suppose $s_3 \in D_{tc}$. Then $s_2 \in D_{tc}$, because $s_0, s_1 \in D_{tc}$. However in this situation $D_{tc} - \{s_0\}$ is a TDS of T of cardinality smaller than $\gamma_t(T)$, a contradiction. Thus $s_3 \notin D_{tc}$ and for this reason s_3 is not a support vertex. If $s_2 \in D_{tc}$, then again $D_{tc} - \{s_0\}$ is a TDS of T, a contradiction. Hence $s_2 \notin D_{tc}$, either.

If $d_T(s_3) = 2$, then $s_4 \in D_{tc}$ and $\gamma_{tc}(T) = n(T) - 2$. Since $\operatorname{diam}(T) \geq 5$ and $\gamma_t(T) = n(T) - 2$, Observations 1 and 2 imply that T is a path P_6. Of course $P_6 \in \mathcal{T}$, so we assume that $d_T(s_3) > 2$.

Denote $T' = T - \{s_0, s_1, s_2\}$. Certainly $\gamma_{tc}(T') \leq \gamma_{tc}(T) - 2$, because $D_{tc} - \{s_0, s_1\}$ is a TCDS of T'. On the other hand, any $\gamma_t(T')$-set may be extended to a TDS of T by adding to it s_0 and s_1, so $\gamma_t(T) \leq \gamma_t(T') + 2$. In this way

$$\gamma_t(T) - 2 \leq \gamma_t(T') \leq \gamma_{tc}(T') \leq \gamma_{tc}(T) - 2 = \gamma_t(T) - 2.$$

Consequently, we must have equalities throughout this inequality chain. In particular, T' is a (γ_t, γ_{tc})-tree and by the induction hypothesis, $T' \in \mathcal{T}$. As s_3 is not a support vertex nor a leaf in T', we conclude that T may be obtained from T' by operation $\mathcal{O}$. □

As an immediate consequence of Lemmas 6 and 7 we have the following characterization of (γ_t, γ_{tc})-trees.

Theorem 8. *A tree T is a (γ_t, γ_{tc})-tree if and only if T belongs to the family $\mathcal{T}$.*

3. Lower bound for the total outer-connected numbers of trees

In this section we give a lower bound for the total outer-connected domination numbers of trees and we characterize the extremal trees.

Theorem 9. *If $T \neq K_{1,k}$ is a tree with at least 3 vertices or $T = K_2$, then $\gamma_{tc}(T) \geq \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$.*

Proof. Let $T \neq K_{1,k}$ be a tree with at least 3 vertices or $T = K_2$. We consider three cases.

Case 1. If $\gamma_{tc}(T) = n(T)$, then $T = K_2$ and the inequality $\gamma_{tc}(T) \geq \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$ holds.

Case 2. If $\gamma_{tc}(T) = n(T) - 1$, then $n(T) \geq 3$ and since T is not a star we conclude that $n(T) \geq n_1(T) + 2$. Hence $\gamma_{tc}(T) = n(T) - 1 = \frac{2n(T)-2}{2} \geq \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$.

Case 3. If $\gamma_{tc}(T) \leq n(T) - 2$, then let D_{tc} be a $\gamma_{tc}(T)$-set. By Proposition 3, $J(T) \subseteq D_{tc}$. Since $T[V(T) - D_{tc}]$ is connected, each vertex of D_{tc}, which is not a leaf, dominates at most one vertex of $V(T) - D_{tc}$. For this reason, $\gamma_{tc}(T) \geq \left\lceil \frac{n(T)-n_1(T)}{2} \right\rceil + n_1 = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$. □

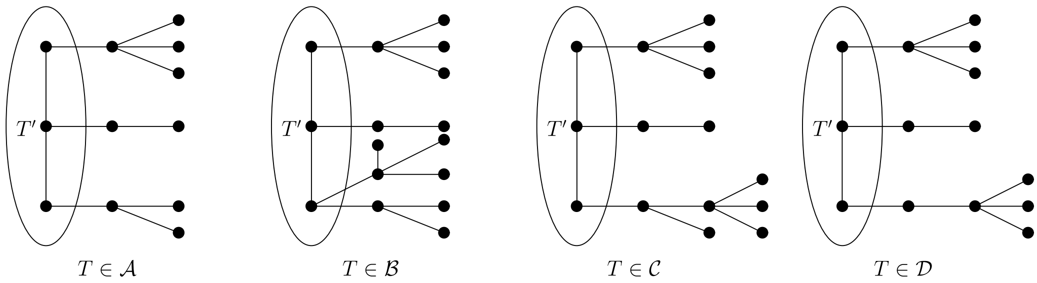

Fig. 3. Trees belonging to the family $\mathcal{R}$.

Now we characterize all trees T for which $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$. For this reason we define $\mathcal{A}$ to be the family of all trees obtained from a tree T' with $V(T') = \{v_1, \dots, v_{n(T')}\}$, $n(T') \geq 2$ by attaching to each vertex of T' a star K_{1,m_i}, $i = 1, \dots, n(T')$, where v_i is joined to a vertex of degree m_i of K_{1,m_i} (see Fig. 3).

Let $\mathcal{B}$ be the family of all trees obtained from a tree $T \in \mathcal{A}$ by attaching a star $K_{1,m}$ to exactly one vertex of $V(T) - J(T)$, say v, by adding the edge joining v to the vertex of degree m of the star $K_{1,m}$ (see Fig. 3).

Let $\mathcal{C}$ be the family of all trees obtained from a tree T' with $V(T') = \{v_1, \dots, v_{n(T')}\}$, $n(T') \geq 2$, by attaching to $n(T') - 1$ vertices of T' stars K_{1,m_i}, $i = 1, \dots, n(T') - 1$, where v_i is joined to a vertex of degree m_i in K_{1,m_i}, and attaching a caterpillar T_c, $\operatorname{diam}(T_c) = 3$, where $v_{n(T')}$ is joined to a support vertex of T_c (see Fig. 3).

Let $\mathcal{D}$ be the family of all trees obtained from a tree T' with $V(T') = \{v_1, \dots, v_{n(T')}\}$, $n(T') \geq 2$, by attaching to $n(T') - 1$ vertices of T' stars K_{1,m_i}, $i = 1, \dots, n(T') - 1$, where v_i is joined to a vertex of degree m_i in K_{1,m_i}, and attaching a star $K_{1,m}$, $m \geq 2$, where $v_{n(T')}$ is joined to a leaf of $K_{1,m}$ (see Fig. 3).

Define $\mathcal{E}$ to be the family of all caterpillars of diameter 3 or 4.

Finally, let $\mathcal{R} = \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \{K_2\}$.

Theorem 10. *If T is a tree with at least 2 vertices, then $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$ if and only if T belongs to the family $\mathcal{R}$.*

Proof. First we prove that if $T \in \mathcal{R}$, then $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$.

If $\mathcal{T} \in \mathcal{A} \cup \mathcal{B} \cup \mathcal{C}$, then certainly $J(T)$ is a TCDS set of T and hence $\gamma_{tc}(T) \leq n_S(T) + n_1(T)$. Moreover, if $T \in \mathcal{A}$, then $n_S(T) = |V(T) - J(T)|$ and if $T \in \mathcal{B} \cup \mathcal{C}$, then $n_S(T) = |V(T) - J(T)| + 1$. Thus

$$\begin{aligned} 2\gamma_{tc}(T) &\leq 2n_S(T) + 2n_1(T) \\ &\leq |V(T) - J(T)| + n_S(T) + 2n_1(T) + 1 \\ &\leq n(T) + n_1(T) + 1. \end{aligned}$$

Therefore, $\gamma_{tc}(T) \leq \left\lfloor \frac{n(T)+n_1(T)+1}{2} \right\rfloor = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$ and Theorem 9 implies that $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$.

If $\mathcal{T} \in \mathcal{D}$, then it is possible to see that $\gamma_{tc}(T) \leq n_S(T) + n_1(T) + 1$ and $n_S(T) \leq |V(T) - J(T)| - 1$. Thus

$$\begin{aligned} 2\gamma_{tc}(T) &\leq 2n_S(T) + 2n_1(T) + 2 \\ &\leq |V(T) - J(T)| + n_S(T) + 2n_1(T) + 1 \\ &\leq n(T) + n_1(T) + 1. \end{aligned}$$

Therefore, $\gamma_{tc}(T) \leq \left\lfloor \frac{n(T)+n_1(T)+1}{2} \right\rfloor = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$ and Theorem 9 implies that $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$.

If $T \in \mathcal{E}$ is a caterpillar of diameter 3 or 4, then $n(T) = n_1(T)+2$ or $n(T) = n_1(T)+3$, respectively. Since $\gamma_{tc}(T) = n(T)-1$ for every tree $T \in \mathcal{E}$, the result follows.

If $T = K_2$, then obviously $\gamma_{tc}(T) = 2 = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$.

Now let T be a tree such that $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil$ and denote by D_{tc} a $\gamma_{tc}(T)$-set.

Case 1. If $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil = n(T)$, then obviously $T = K_2 \in \mathcal{R}$.

Case 2. If $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil = n(T) - 1$, then $n(T) = n_1(T) + 2$ or $n(T) = n_1(T) + 3$. Therefore, T is a caterpillar of diameter 3 or 4 and thus $T \in \mathcal{R}$.

Case 3. If $\gamma_{tc}(T) = \left\lceil \frac{n(T)+n_1(T)}{2} \right\rceil \leq n(T) - 2$, then by Proposition 3, $J(T) \subseteq D_{tc}$. Let a be the number of connected components of the subgraph induced by D_{tc}. Each connected component of $T[D_{tc}]$ contains a support vertex of T, so

$$a \leq n_S(T) \leq \gamma_{tc}(T) - n_1(T) = \left\lceil \frac{n(T) - n_1(T)}{2} \right\rceil.$$

Since the graph $T[V(T) - D_{tc}]$ is connected, at most one vertex from each connected component of $T[D_{tc}]$ is adjacent to a vertex of $V(T) - D_{tc}$. Hence,

$$a \geq n(T) - \gamma_{tc}(T) = n(T) - \left\lceil \frac{n(T) + n_1(T)}{2} \right\rceil = \left\lfloor \frac{n(T) - n_1(T)}{2} \right\rfloor.$$

Therefore

$$\left\lfloor \frac{n(T) - n_1(T)}{2} \right\rfloor \leq a \leq n_S(T) \leq \left\lceil \frac{n(T) - n_1(T)}{2} \right\rceil$$

and we consider four subcases.

Subcase 3.1. $\left\lfloor \frac{n(T)-n_1(T)}{2} \right\rfloor = a = n_S(T) = \left\lceil \frac{n(T)-n_1(T)}{2} \right\rceil$. Then $J(T) = D_{tc}$ and each connected component of $T[D_{tc}]$ contains exactly one support vertex of T. Moreover, each support vertex dominates exactly one vertex of $V(T) - J(T)$ and each vertex of $V(T) - J(T)$ is adjacent to exactly one support vertex. Therefore T may be obtained from a tree $T' = T[V(T) - J(T)]$ with $V(T') = V(T) - J(T) = \{v_1, \ldots, v_{n(T')}\}$, $n(T') \geq 2$, by attaching to each vertex of T' a star K_{1,m_i}, $i = 1, \ldots, n(T')$, where v_i is joined to a vertex of degree m_i in K_{1,m_i}. Thus, $T \in \mathcal{A} \subseteq \mathcal{R}$.

Subcase 3.2. $\left\lfloor \frac{n(T)-n_1(T)}{2} \right\rfloor < a = n_S(T) = \left\lceil \frac{n(T)-n_1(T)}{2} \right\rceil$. Then $J(T) = D_{tc}$ and each connected component of $T[D_{tc}]$ contains exactly one support vertex of T. Moreover, each support vertex dominates exactly one vertex of $V(T) - J(T)$ and $n(T) - \gamma_{tc}(T) - 1$ vertices of $V(T) - J(T)$ is adjacent to exactly one support vertex and one vertex of $V(T) - J(T)$, say v, is adjacent to exactly two support vertices, say x and y. Therefore, T may be obtained from a tree $T' \in \mathcal{A}$, $T' = T[(V(T) - N_T[y]) \cup \{v\}]$, by attaching a star $K_{1,m}$ to v by adding the edge joining v to the vertex of degree m of the star $K_{1,m}$ (that is y). Thus, $T \in \mathcal{B} \subseteq \mathcal{R}$.

Subcase 3.3. $\left\lfloor \frac{n(T)-n_1(T)}{2} \right\rfloor = a < n_S(T) = \left\lceil \frac{n(T)-n_1(T)}{2} \right\rceil$. Then $J(T) = D_{tc}$. Moreover, $a - 1$ connected components of $T[D_{tc}]$ contains exactly one support vertex of T and one connected component of $T[D_{tc}]$ contains exactly two support vertices of T. Further, each vertex of $V(T) - J(T)$ is adjacent to exactly one support vertex and since T is connected, we conclude that exactly one support vertex from each connected component of $T[D_{tc}]$ dominates a vertex of $V(T) - D_{tc}$. Therefore T may be obtained from a tree $T' = T[V(T) - J(T)]$ with $V(T') = \{v_1, \ldots, v_{n(T')}\}$, $n(T') \geq 2$, by attaching to $n(T') - 1$ vertices of T' stars K_{1,m_i}, $i = 1, \ldots, n(T') - 1$, where v_i is joined to a vertex of degree m_i in K_{1,m_i}, and attaching caterpillar T_c, $\operatorname{diam}(T_c) = 3$, where $v_{n(T')}$ is joined to a support vertex of T_c. Thus, $T \in \mathcal{D} \subseteq \mathcal{R}$.

Subcase 3.4. $\left\lfloor \frac{n(T)-n_1(T)}{2} \right\rfloor = a = n_S(T) < \left\lceil \frac{n(T)-n_1(T)}{2} \right\rceil$. Then $D_{tc} - J(T)$ contains exactly one vertex, and each connected component of $T[D_{tc}]$ contains exactly one support vertex of T. Moreover, each support vertex dominates exactly one vertex of $V(T) - J(T)$ and each vertex of $V(T) - J(T)$ is adjacent to exactly one support vertex. Therefore T may be obtained from a tree $T' = T[V(T) - D_{tc}]$ with $V(T') = \{v_1, \ldots, v_{n(T')}\}$, $n(T') \geq 2$, by attaching to $n(T') - 1$ vertices of T' stars K_{1,m_i}, $i = 1, \ldots, n(T') - 1$, where v_i is joined to a vertex of degree m_i in K_{1,m_i}, and attaching a star $K_{1,m}$, $m \geq 2$, where $v_{n(T')}$ is joined to a leaf of $K_{1,m}$. Thus, $T \in \mathcal{C} \subseteq \mathcal{R}$. $\square$

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