J. Fixed Point Theory Appl. 16 (2014) 301–311
DOI 10.1007/s11784-015-0212-9
Published online January 21, 2015

Journal of Fixed Point Theory and Applications

Two families of infinitely many homoclinics for singular strong force Hamiltonian systems

Marek Izydorek and Joanna Janczewska

To Professor Andrzej Granas

Abstract. We are concerned with a planar autonomous Hamiltonian system $\ddot{q}+\nabla V(q)=0$, where a potential $V\colon \mathbb{R}^2\setminus\{\xi\}\to\mathbb{R}$ has a single well of infinite depth at a point ξ and a unique strict global maximum 0 at a point a. Under a strong force condition around the singularity ξ, via minimization of an action integral and using a shadowing chain lemma together with simple geometrical arguments, we prove the existence of infinitely many homotopy classes of $\pi_1(\mathbb{R}^2\setminus\{\xi\})$ containing at least two geometrically distinct homoclinic (to a) solutions.

Mathematics Subject Classification. Primary 34C37; Secondary 70H05.

Keywords. Homoclinic orbit, homotopy class, shadowing chain lemma, singular Hamiltonian system, strong force, rotation number (winding number).

1. Introduction

In this paper we are concerned with the second order Hamiltonian system

$$\ddot{q}+\nabla V(q)=0, \tag{HS}$$

where $\ddot{} = \frac{d^2}{dt^2}$, $q\in\mathbb{R}^2$ and ∇V denotes the gradient of a potential V. We denote by $|\cdot|$ the norm in $\mathbb{R}^2$ induced by the standard inner product $(\cdot,\cdot)$. Throughout the paper we assume that the potential V satisfies the following conditions:

- (V_1) there is $\xi\in\mathbb{R}^2$ such that $V\in C^{1,1}(\mathbb{R}^2\setminus\{\xi\},\mathbb{R})$ and $\lim_{x\to\xi}V(x)=-\infty$,
- (V_2) there are a neighbourhood $\mathcal{N}\subset\mathbb{R}^2$ of the point ξ and a function $U\in C^1(\mathcal{N}\setminus\{\xi\},\mathbb{R})$ such that $|U(x)|\to\infty$ as $x\to\xi$ and $|\nabla U(x)|^2\le -V(x)$ for all $x\in\mathcal{N}\setminus\{\xi\}$,
- (V_3) $V(x)\le 0$ and V has a unique maximum at a point $a\in\mathbb{R}^2\setminus\{\xi\}$, $V(a)=0$,

(V_4) there is a negative constant V_0 such that $\limsup_{|x|\to\infty} V(x) \leq V_0$.

Under the above assumptions, applying a variational approach we study the existence and multiplicity of (nonstationary) homoclinic solutions of (HS) which, as $t \to \pm\infty$, are asymptotic to the stationary point a and omit the singularity ξ. In other words, we are looking for solutions such that $q(t) \neq \xi$ for all $t \in \mathbb{R}$, $q(t) \to a$ and $\dot{q}(t) \to 0$ as $t \to \pm\infty$.

Condition (V_2), known in the literature as the strong force condition or Gordon's condition, was introduced by W. B. Gordon in [7]. It governs the rate at which $V(x) \to -\infty$ as $x \to \xi$ and holds, for example, if $\alpha \geq 2$ for $V(x) = -|x-\xi|^{-\alpha}$ nearby ξ. If $V\colon \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfies (V_2), then $\nabla V : \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}^2$ is called a strong force, and (HS) is said to be a strong force system. Moreover, (V_2) implies that the system (HS) does not possess solutions in $W^{1,2}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)$, entering the singular point ξ in finite time. Gordon's condition excludes the gravitational case and leads to the disclosure of certain differences between the behaviour of strong force systems and gravitational ones.

Condition (V_4) can be replaced by somewhat weaker assumption:

(V_4') $\lim_{|x|\to\infty} |x|^2 V(x) = -\infty$.

In [5], under assumptions (V_1)–(V_4') and some geometric condition $(\star)$ on V due to Bolotin (see [2]), Caldiroli and Jeanjean proved the existence of infinitely many homoclinics of (HS), each one being characterized by a distinct winding number around the singularity ξ.

The aim of this work is to show by the use of minimization arguments that under hypotheses (V_1)–(V_4) and somewhat stronger geometric condition than Bolotin's one, there are infinitely many homotopy classes of $\pi_1(\mathbb{R}^2 \setminus \{\xi\})$ containing at least two geometrically distinct homoclinic solutions of (HS).

The existence of homoclinic orbits is an important problem in the study of the behaviour of dynamical systems. Their existence may give the horseshoe chaos (see, for example, [18] and the references therein). The presence of infinitely many geometrically distinct homoclinic or heteroclinic orbits is an indication of nonintegrability and chaotic behaviour for the system (HS) (see [2, 3]).

There have been several other papers in recent years which use variational methods to find homoclinic or heteroclinic orbits of autonomous strong force Hamiltonian systems (see [1, 4, 6, 9, 11, 12, 19]) and periodically forced ones (see [10, 16]). Moreover, Rabinowitz obtained homoclinic and multibump solutions for both periodically and almost periodically forced singular Hamiltonian systems (see [14, 15, 17]).

2. Multiplicity results

At the beginning we set up notation and terminology. It is well known that the Sobolev space

$$E = \left\{ q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^2)\colon \int_{-\infty}^{\infty} |\dot{q}(t)|^2\, dt < \infty \right\}$$

equipped with the norm given by

$$\|q\|^2 = \int_{-\infty}^{\infty} |\dot{q}(t)|^2\, dt + |q(0)|^2$$

is a Hilbert space. Let

$$\alpha_\varepsilon = \inf\{-V(x)\colon x \notin B_\varepsilon(a)\},$$

where $0 < \varepsilon \le \frac{1}{3}|a-\xi|$ and $B_\varepsilon(a)$ denotes the ball of radius ε around a. By (V_1), (V_3) and (V_4) it follows that $\alpha_\varepsilon > 0$. For $q \in E$, set

$$I(q) = \int_{-\infty}^{\infty} \left(\frac{1}{2}|\dot{q}(t)|^2 - V(q(t))\right) dt.$$

Let

$$A = \left\{q \in E\colon \lim_{t\to\pm\infty} q(t) = a,\, q(\mathbb{R}) \subset \mathbb{R}^2 \setminus \{\xi\}\right\}.$$

Lemma 2.1. *Suppose that $q \in E$ and $q(t) \notin B_\varepsilon(a)$ for each $t \in \bigcup_{i=1}^k [r_i, s_i]$, where $[r_i, s_i] \cap [r_j, s_j] = \emptyset$ for $i \neq j$. Then*

$$I(q) \ge \sqrt{2\alpha_\varepsilon} \sum_{i=1}^k \big|q(s_i) - q(r_i)\big|.$$

An easy proof of this lemma can be found in [8].

To shorten notation, $q(\pm\infty) = \lim_{t\to\pm\infty} q(t)$. Applying Lemma 2.1 one can prove that if $I(q) < \infty$, then $q \in L^\infty(\mathbb{R}, \mathbb{R}^2)$ and $q(\pm\infty) = a$ (cf. [8, Corollary 2.2 and Lemma 2.4]). Furthermore, if $[t, s]$ is an interval such that $q([t, s]) \subset \mathcal{N} \setminus \{\xi\}$, then by (V_2),

$$\big|U(q(s))\big| - \big|U(q(t))\big| \le \sqrt{2}\, I(q),$$

which implies that $q(t) \neq \xi$ for $t \in \mathbb{R}$ (cf. [16, eq. (2.21)]). Thus, if $I(q) < \infty$, then $q \in A$. Consequently, q describes a closed curve in $\mathbb{R}^2 \setminus \{\xi\}$ that starts and ends at a. Hence its homotopy class $[q]$ represents an element of the fundamental group $\pi_1(\mathbb{R}^2 \setminus \{\xi\})$.

Let us remind that two elements $q_0, q_1 \in A$ are homotopic if and only if there exists a continuous map $h\colon [0,1] \to A$ such that

$$h(0) = q_0 \quad \text{and} \quad h(1) = q_1.$$

The rotation number (the winding number) $\mathrm{rot}_\xi(q)$ of q around ξ is constant on every connected component of A and induces an isomorphism

$$\mathrm{rot}_*\colon \pi_1(\mathbb{R}^2 \setminus \{\xi\}) \to \mathbb{Z}, \quad \mathrm{rot}_*([q]) = \mathrm{rot}_\xi(q).$$

Equivalently, A is a sum of its path-connected components labeled by the integers.

We define the family $\mathcal{F}$ as follows. A set $Z \subset A$ is a member of $\mathcal{F}$ if and only if

- for each $q \in Z$ and for each $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ there exists $\delta > 0$ such that if $s \in (-\delta, \delta)$, then $q + s\psi \in Z$.

Let us remark that if q is a minimizer of I on a set $Z \in \mathcal{F}$, then

$$\frac{d}{ds} I(q+s\psi)|_{s=0} = 0 = \int_{-\infty}^{\infty} \big((\dot{q}(t), \dot{\psi}(t)) - \big(\nabla V(q(t)), \psi(t)\big)\big) dt,$$

and consequently, q is a weak solution of (HS). Analysis similar to that in the proof of Proposition 3.18 in [13] shows that q is a classical solution of (HS). Finally, using (HS), (V_1) and (V_3) as in [12] gives $\dot{q}(\pm\infty) = 0$.

Let $l(s)$ be the line through a and ξ parameterized by $s \in \mathbb{R}$ in such a way that a and ξ correspond to $s = 0$ and $s = 1$, respectively. The line l divides $\mathbb{R}^2$ into two half-planes $\Pi^{\pm}$. To be more precise, if $(\vec{a\xi}, \vec{\mathbf{e}})$ is a positively oriented orthogonal basis in $\mathbb{R}^2$, then $\vec{\mathbf{e}} \in \Pi^+$.

Let

$$G = \{q \in A \colon q(t) \neq a \text{ for all } t \in \mathbb{R}\},$$

the set of curves in A that do not achieve a in finite time.

Definition 2.1. We say that $q_0, q_1 \in G$ are homotopic (in G) if there is a continuous map $h \colon [0,1] \to G$ with $h(0) = q_0$ and $h(1) = q_1$.

In other words, q_0 and q_1 are homotopic in G if and only if they belong to the same path-connected component of G. The homotopy class of $q \in G$ is denoted by $[\![q]\!]$, and Γ is the set of homotopy classes. The inclusion $\iota \colon G \to A$ induces a surjective map

$$\iota_* \colon \Gamma \to \pi_1(\mathbb{R}^2 \setminus \{\xi\}), \quad \iota_*([\![q]\!]) = [q].$$

In fact, for every $[q] \in \pi_1(\mathbb{R}^2 \setminus \{\xi\})$ the inverse image $\iota_*^{-1}([q])$ contains infinitely many elements. We are going to describe the set Γ.

Lemma 2.2. *Every homotopy class $\gamma \in \Gamma$ can be represented by $q \in G$ that has at most finitely many intersection points with the line l.*

Proof. Let $\gamma = [\![q]\!]$ for some loop $q \in G$. Choose $T \in \mathbb{R}$ such that

$$|q(t) - a| < |a - \xi| \quad \text{for } |t| > T.$$

By standard transversality (or simplicial approximation) arguments there is a perturbation q_0 of q in G that has at most finitely many intersection points with the line l on the interval $[-T, T]$ and $q_0(-T), q_0(T) \notin l$. Let us introduce the polar coordinate system in $\mathbb{R}^2$ with the pole a and the polar axis l whose orientation agrees with the orientation of the plane. In this coordinate system one has $q_0(t) = (r(t)\cos\varphi(t), r(t)\sin\varphi(t))$. Clearly, there is no uniqueness of a function $\varphi(t)$. Since $q_0(t)$ is continuous we can assume that $\varphi(t)$ is continuous. Furthermore, $r(t) > 0$ for every $t \in \mathbb{R}$. Consider the restriction of $q_0(t)$ to the interval $t \geq T$. Define a map $H \colon [T, \infty) \times [0,1] \to \mathbb{R}^2$,

$$H(t,s) = \big(r(t)\cos\big((1-s)\varphi(t) + s\varphi(T)\big), r(t)\sin\big((1-s)\varphi(t) + s\varphi(T)\big)\big).$$

Thus, $H(t,0) = q_0(t)$, and $H(t,1) = (r(t)\cos(\varphi(T)), r(t)\sin(\varphi(T)))$ has no crossing points with the line l. Moreover, if we put $q_s(t) = H(t,s)$, then

$$\int_T^{\infty} |\dot{q}_s(t)|^2\, dt = \int_T^{\infty} \dot{r}(t)^2 + r(t)^2(1-s)^2\dot{\varphi}(t)^2\, dt \leq \int_T^{\infty} |\dot{q}_0(t)|^2\, dt < \infty.$$

Consequently,

$$Q_s(t) = \begin{cases} q_0(t) & \text{if } t \le T, \\ q_s(t) & \text{if } t > T, \end{cases}$$

is a homotopy in G. The case $t < -T$ is analogous. □

Given a homotopy class $[\![q]\!]$, assume that q has a minimal number, $k > 0$, of crossing points with the line l. Thus there are $t_1 < t_2 < \cdots < t_k$ such that $q(t_i) = l(s_i)$ for certain $s_i \in \mathbb{R}$, $i = 1, \ldots, k$. We associate with q a word ω of length k as follows. If q crosses the line l at time t_i, leaving Π^- and entering Π^+, then at the ith place in ω we will write

$$\begin{array}{ll} u & \text{if } s_i > 1, \\ v & \text{if } 0 < s_i < 1, \\ w & \text{if } s_i < 0. \end{array}$$

If q crosses l living Π^+ and entering Π^-, then we use letters $\bar{u}$, $\bar{v}$, $\bar{w}$, respectively. If $[\![q]\!] \in \Gamma$, then the corresponding word ω has the following properties:

- ω begins and ends at the letter u (with or without a bar, i.e., $\bar{u}$, u),
- two consecutive letters in ω are never the same,
- every second letter in ω appears with a bar.

The set of words satisfying the above conditions is denoted by Ω. Additionally, a contractible loop is represented by the empty word. For every $\omega \in \Omega$ we define $\bar{\omega} \in \Omega$ as follows. We remove all bars from the word ω. Next we put bars over letters that appear in ω without bars and finally we write letters in the opposite order. For instance, if $\omega = u\bar{w}u\bar{v}u$, then $\bar{\omega} = \bar{u}v\bar{u}w\bar{u}$. It is clear that ω is represented by a loop $q(t)$ if and only if $\bar{\omega}$ is represented by $q(-t)$.

Proposition 2.3. *The procedure described above defines a bijection*

$$\mathcal{B}\colon \Omega \to \Gamma.$$

The proof is left to the reader.

Given a word $\omega \in \Omega$ of length k. Assume that the letter u (with or without a bar) appears at ith and jth places in ω and there is no u at places with indices between i and j. We define a derived from ω sequence of words $\omega_1 \cup \omega_2$ as follows. The word ω_1 is a sequence of the first i elements of ω and ω_2 is a sequence of the last $k - j + 1$ elements of ω. Clearly, $\omega_1, \omega_2 \in \Omega$ and the decomposition depends on the choice of i and j. This procedure can be iterated, and any sequence $\omega_1 \cup \cdots \cup \omega_d$ obtained in this way is called a derived from ω sequence of words. Let u appear $u(\omega)$ times and $\bar{u}$ appear $\bar{u}(\omega)$ times in a word ω.

Set

$$\rho_\omega = u(\omega) + \bar{u}(\omega).$$

Consider the composition

$$\mathrm{ind}_\xi = \mathrm{rot}_* \circ \iota_* \circ \mathcal{B}\colon \Omega \to \mathbb{Z}.$$

Proposition 2.4. *For every* $\omega \in \Omega$ *one has* $\mathrm{ind}_\xi(\omega) = u(\omega) - \bar{u}(\omega)$.

Proof. Given $\omega \in \Omega$, choose $q \in G$ such that $\mathcal{B}(\omega) = [\![q]\!]$ and q has a minimal number of crossing points with the line l. Then

$$\mathrm{ind}_\xi(\omega) = \mathrm{rot}_*\left(\iota_*([\![q]\!])\right) = \mathrm{rot}_*([q]) = \mathrm{rot}_\xi(q) = u(\omega) - \bar{u}(\omega). \qquad \square$$

Corollary 2.5. *If $\omega_1 \cup \cdots \cup \omega_d$ is a derived from ω sequence of words, then*

$$\mathrm{ind}_\xi(\omega) = \sum_{i=1}^{d} \mathrm{ind}_\xi(\omega_i) \quad \textit{and} \quad \rho_\omega = \sum_{i=1}^{d} \rho_{\omega_i}.$$

For each $\omega \in \Omega$, let

$$\Gamma_\omega = \left\{q \in G \colon [\![q]\!] = \mathcal{B}(\omega)\right\},$$

a path-connected component of Γ. It is easily seen that for every $\omega \in \Omega$, Γ_ω is a member of the family $\mathcal{F}$. Define

$$\lambda_\omega = \inf\{I(q) \colon q \in \Gamma_\omega\}.$$

Clearly, if $\omega_1 \cup \cdots \cup \omega_d$ is a derived from ω sequence of words, then

$$\lambda_\omega \le \sum_{i=1}^{d} \lambda_{\omega_i}.$$

In particular, since $\lambda_\omega = \lambda_{\bar{\omega}}$, one has

$$\lambda_\omega \le \rho_\omega \lambda_u.$$

Corollary 2.6. *Let $\omega_1 \cup \cdots \cup \omega_d$ be a derived from ω sequence of words. If $\lambda_\omega = \rho_\omega \lambda_u$, then $\lambda_{\omega_i} = \rho_{\omega_i} \lambda_u$ for every $i = 1, 2, \ldots, d$.*

Analogously to the proof of Lemma 3.2 in [9] one proves the following version of the shadowing chain lemma.

Theorem 2.7 (Shadowing chain lemma). *Let $\omega \in \Omega$. Under conditions* (V_1)–(V_4), *there are a derived from ω sequence $\omega_1 \cup \cdots \cup \omega_d$ and homoclinic solutions $Q_{\omega_i} \in \Gamma_{\omega_i}$, $i = 1, \ldots, d$, of the Hamiltonian system* (HS) *such that*

$$\lambda_\omega = \sum_{i=1}^{d} I(Q_{\omega_i}) = \sum_{i=1}^{d} \lambda_{\omega_i}.$$

Assume that

($\star$) there exist $T \in (0, \infty)$ and $p \in W^{1,2}([0,T], \mathbb{R}^2 \setminus \{\xi\})$ such that $p(0) = p(T)$, $\mathrm{rot}_\xi(p) = 1$ and

$$\int_0^T \left(\frac{1}{2}|\dot{p}|^2 - V(p)\right) dt < \lambda_u.$$

This geometric condition has been introduced by Bolotin [2].

Under conditions (V_1)–(V_4) and ($\star$) it has been proved in [5] (cf. Theorem 1.1) that

there is $k_0 \in \mathbb{N}$ such that for every $k > k_0$ there exists a homoclinic solution $Q_k \in A$ of (HS) *with* $\mathrm{rot}_\xi(Q_k) = k$.

Let us introduce somewhat stronger Bolotin's type assumption on the geometry of V. Namely, assume that

(B) for $i = 1, 2$ there exist $T_i \in (0, \infty)$ and $p_i \in W^{1,2}([0, T_i], \mathbb{R}^2 \setminus \{a, \xi\})$ such that $p_i(0) = p_i(T_i)$ and

$(B1)$ $\mathrm{rot}_a(p_1) = 0$, $\mathrm{rot}_\xi(p_1) = 1$ and

$$\int_0^{T_1} \left(\frac{1}{2}|\dot{p}_1|^2 - V(p_1)\right) dt < \lambda_u,$$

$(B2)$ $\mathrm{rot}_a(p_2) = \mathrm{rot}_\xi(p_2) = 1$ and

$$\int_0^{T_2} \left(\frac{1}{2}|\dot{p}_2|^2 - V(p_2)\right) dt < \lambda_u.$$

For each $n \in \mathbb{N}$, define two sequences of words in Ω as follows:

$$\mu_n = (u\bar{v})^{n-1}u \quad \text{and} \quad \nu_n = (u\bar{w})^{n-1}u.$$

Notice that $\rho_{\mu_n} = \rho_{\nu_n} = n$, hence $\lambda_{\mu_n} \leq n\lambda_u$ and $\lambda_{\nu_n} \leq n\lambda_u$ for $n \in \mathbb{N}$.

Set

$$m_0 = \sup\{n \in \mathbb{N} \colon \lambda_{\mu_n} = n\lambda_u\} \quad \text{and} \quad n_0 = \sup\{n \in \mathbb{N} \colon \lambda_{\nu_n} = n\lambda_u\}.$$

One can easily prove the following proposition.

Proposition 2.8. *If (B) is satisfied, then both numbers m_0 and n_0 are finite.*

We can now formulate our main result.

Theorem 2.9. *Let $V \colon \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfy (V_1)–(V_4) and (B).*

- *If $(B1)$ holds, then for every $k > m_0$ there exists $P_k \in \Gamma_{\mu_k}$ such that $I(P_k) = \lambda_{\mu_k}$. Moreover, P_k is a homoclinic solution of* (HS).
- *If $(B2)$ holds, then for every $k > n_0$ there exists $Q_k \in \Gamma_{\nu_k}$ such that $I(Q_k) = \lambda_{\nu_k}$. Moreover, Q_k is a homoclinic solution of* (HS).

We prove both cases $(B1)$ and $(B2)$, following ideas of the proof of Lemma 4.2 in [5]. That proof relies on the following lemma due to Rabinowitz (cf. [16, Proposition 3.41]).

Lemma 2.10. *Under hypotheses (V_1)–(V_4), if $q \in \Gamma_\omega$, with $\mathrm{ind}_\xi(\omega) \geq 2$ and $I(q) = \inf\{I(q) \colon q \in \Gamma_\omega\}$, then there exist $t, s \in \mathbb{R}$ such that $t < s$, $q(t) = q(s)$ and* $\mathrm{rot}_\xi(q_{|[t,s]}) = 1$.

In fact, every loop q satisfying $\mathrm{rot}_\xi(q) \geq 2$ contains a subloop with the rotation number equal to 1. The proof of this topological property will be given in the appendix.

Sketch of the proof of Theorem 2.9. Let us observe that in case $(B1)$ a closed orbit $\bar{P}$ existing by Proposition A.3 (see the appendix) can be chosen in such a way that $\mathrm{rot}_a(\bar{P}) = 0$, whereas in case $(B2)$, there is a closed orbit $\bar{Q}$ such that $\mathrm{rot}_a(\bar{Q}) = 1$. Lemma 4.2 of [5] implies the following inequalities:

$$\lambda_u = \frac{1}{2}\lambda_{\mu_2} = \cdots = \frac{1}{m_0}\lambda_{\mu_{m_0}} > \frac{1}{m_0+1}\lambda_{\mu_{m_0+1}} > \cdots \tag{2.1}$$

and

$$\lambda_u = \frac{1}{2}\lambda_{\nu_2} = \cdots = \frac{1}{n_0}\lambda_{\nu_{n_0}} > \frac{1}{n_0+1}\lambda_{\nu_{n_0+1}} > \cdots. \tag{2.2}$$

For a fixed $n > m_0$ consider a derived from μ_n sequence of words

$$\omega_1 \cup \cdots \cup \omega_d, \quad d > 1.$$

Then, for any $i = 1, \ldots, d$, we have $\omega_i = \mu_{j_i}$ for some $j_i < n$ and $\sum_{i=1}^{d} j_i = n$. Thus, by (2.1),

$$\sum_{i=1}^{d} \lambda_{\omega_i} = \sum_{i=1}^{d} \lambda_{\mu_{j_i}} > \sum_{i=1}^{d} \frac{j_i}{n} \lambda_{\mu_n} = \lambda_{\mu_n}.$$

Now the existence of a homoclinic solution $P_n \in \Gamma_{\mu_n}$ of (HS) follows from Theorem 2.7. Similar argumentation applied to ν_n, $n > n_0$, together with (2.2) gives the existence of a homoclinic solution $Q_n \in \Gamma_{\nu_n}$ of (HS). □

Remark 2.11. Each homotopy class $[q] \in \pi_1(\mathbb{R}^2 \setminus \{\xi\})$ with a sufficiently large rotation contains at least two geometrically distinct homoclinic solutions of (HS).

Observe that one family of solutions is represented by words containing u's and v's, whereas another family is represented by words containing u's and w's. It would be interesting to know if, except the above quite specific families of solutions, there exist solutions represented by more "complicated" words, in particular, words containing three letters. To this purpose let us modify condition (B). Assume that

(B') for $i = 1, 2$ there exist $T_i \in (0, \infty)$ and $p_i \in W^{1,2}([0, T_i], \mathbb{R}^2 \setminus \{a, \xi\})$ such that $p_1(0) = p_1(T_1) = p_2(0) = p_2(T_2)$ and

$(B'1)$ $\mathrm{rot}_a(p_1) = 0$, $\mathrm{rot}_\xi(p_1) = 1$ and

$$\int_0^{T_1} \left(\frac{1}{2} |\dot{p}_1|^2 - V(p_1) \right) dt < \lambda_u,$$

$(B'2)$ $\mathrm{rot}_a(p_2) = \mathrm{rot}_\xi(p_2) = 1$ and

$$\int_0^{T_2} \left(\frac{1}{2} |\dot{p}_2|^2 - V(p_2) \right) dt < \lambda_u.$$

Define two sequences of words in Ω as follows. For $n \in \mathbb{N}$,

- $\tau_n = u\bar{w}u\bar{v}\ldots u$ consists of $2n-1$ letters, u appears at every second place, $\bar{w}$ and $\bar{v}$ appear at every fourth place,
- $\sigma_n = u\bar{v}u\bar{w}\ldots u$ is obtained from τ_n by interchanging $\bar{w}$ with $\bar{v}$.

Clearly, $\rho_{\tau_n} = \rho_{\sigma_n} = n$, hence $\lambda_{\tau_n} \le n\lambda_u$ and $\lambda_{\sigma_n} \le n\lambda_u$ for $n \in \mathbb{N}$.

Set

$$k_\tau = \sup\{n \colon \lambda_{\tau_n} = n\lambda_u\} \quad \text{and} \quad k_\sigma = \sup\{n \colon \lambda_{\sigma_n} = n\lambda_u\}.$$

Proposition 2.12. *If (B') is satisfied, then both numbers k_τ and k_σ are finite.*

The proof is straightforward.

Theorem 2.13. *Let $V \colon \mathbb{R}^2 \setminus \{\xi\} \to \mathbb{R}$ satisfy (V_1)–(V_4) and (B'). Let $2 \le k_\tau \le k_\sigma$. In addition to the families of solutions given by Theorem 2.9, there exists $Q \in \Lambda_{\tau_k}$ for which $I(Q) = \lambda_{\tau_k}$, where $k = k_\tau + 1$. Moreover, Q is a homoclinic solution of (HS).*

Proof. Set $k = k_\tau + 1$. The inequality $\lambda_{\tau_k} < \rho_{\tau_k}\lambda_u$ follows from the definition of k_τ. It is enough to show that for any derived from τ_k sequence of words $\omega_1 \cup \cdots \cup \omega_d$ with $d > 1$ one has

$$\lambda_{\omega_i} = \rho_{\omega_i}\lambda_u. \tag{2.3}$$

Then the conclusion follows from Theorem 2.7. If $\omega_1 \neq u$, then (2.3) follows from Corollary 2.6 applied to the word τ_k. If $\omega_1 = u$, then (2.3) is a consequence of Corollary 2.6 applied to σ_{k_σ}. □

Remark 2.14. Clearly, if $2 \le k_\sigma \le k_\tau$, then (HS) possesses a homoclinic solution $P \in \Gamma_{\sigma_k}$, $k = k_\sigma + 1$. Furthermore, if $2 \le k_\sigma = k_\tau$, then (HS) possesses two homoclinic solutions $Q \in \Gamma_{\tau_k}$ and $P \in \Gamma_{\sigma_k}$.

Appendix

Let $\sigma\colon [0,1] \to \mathbb{R}^2 \setminus \{\xi\}$ be a loop; i.e., $q(0) = q(1)$. A point $p \in \sigma([0,1])$ is *a simple crossing point* if there are exactly two numbers $t, s \in [0,1]$ such that $p = \sigma(t) = \sigma(s)$. A loop σ is *regular* if it possesses at most finitely many crossing points each of which is simple. If $\sigma(a) = \sigma(b)$ for some numbers $a, b \in [0,1]$, $a < b$, then $\sigma_{|[a,b]}$ is *a subloop* of σ. Removing a subloop $\sigma_{|[a,b]}$ from the loop σ, we obtain a new loop $\bar\sigma\colon [0, 1-(b-a)] \to \mathbb{R}^2 \setminus \{\xi\}$,

$$\bar\sigma(t) = \begin{cases} \sigma(t) & \text{if } t \le a, \\ \sigma(t+b-a) & \text{if } t > a. \end{cases}$$

We will write $\bar\sigma = \sigma - \sigma_{|[a,b]}$. Our aim is to prove the following proposition.

Proposition A.1. *If q is a loop in $\mathbb{R}^2 \setminus \{\xi\}$ with $\operatorname{rot}_\xi(q) \ge 2$, then there exist $t, s \in [0,1]$ such that $t < s$, $q(t) = q(s)$ and $\operatorname{rot}_\xi(q_{|[t,s]}) = 1$.*

Lemma A.2. *Let $\sigma\colon [0,1] \to \mathbb{R}^2 \setminus \{\xi\}$ be a regular loop with $\operatorname{rot}_\xi \sigma \ge 2$. Then there are $0 < t < s < 1$ such that $\sigma(t) = \sigma(s)$ and $\operatorname{rot}_\xi \sigma = 1$.*

Proof. Let k be a number of crossing points of σ. Thus there are $0 < t_1 < \cdots < t_k < 1$ and a set $\{s_1, \ldots, s_k\} \subset [0,1]$ such that $p_i = \sigma(t_i) = \sigma(s_i)$, for $i = 1, \ldots, k$, are simple crossing points. We do not consider $p_0 = \sigma(0) = \sigma(1)$. If $\operatorname{rot}_\xi \sigma_{|[t_1,s_1]} \le 0$, then the loop $\sigma' = \sigma - \sigma_{|[t_1,s_1]}$ has at most $k-1$ simple crossing points and $\operatorname{rot}_\xi \sigma' \ge 2$. Choose the smallest $t \in [0, 1+a-b]$, $t > 0$, for which there is $s \in [0, 1+a-b]$ such that $\sigma'(t) = \sigma'(s)$. If $\operatorname{rot}_\xi \sigma_{|[t,s]} \le 0$, then we define $\sigma'' = \sigma' - \sigma_{|[t,s]}$ which has at most $k-2$ simple crossing points and satisfies $\operatorname{rot}_\xi \sigma'' \ge 2$. As a consequence of this procedure there exists t_i such that $\operatorname{rot}_\xi \sigma_{|[t_i,s_i]} \ge 1$. Otherwise we end up with a loop that has a rotation number greater than or equal to two and has no crossing points. If $\operatorname{rot}_\xi \sigma_{|[t_i,s_i]} = 1$, then we are done. If it is not the case, we apply the above procedure to the loop $\sigma_1 = \sigma_{|[t_i,s_i]}$ that has at most $k-1$ simple crossing points. Thus σ_1 contains a subloop σ_2 such that $\operatorname{rot}_\xi \sigma_2 \ge 1$ and σ_2 has at most $k-2$ crossing points. This procedure can be repeated at most k times and finally we obtain a loop σ_k such that $\operatorname{rot}_\xi \sigma_k \ge 1$ and σ_k has no crossing points. Hence $\operatorname{rot}_\xi \sigma_k = 1$. □

Proof of Proposition A.1. Choose $n_0 \in \mathbb{N}$ such that

$$\frac{1}{n_0} < \operatorname{dist}\big(\sigma([0,1]),\xi\big).$$

For every $n \geq n_0$ let $\sigma_n\colon [0,1] \to \mathbb{R}^2 \setminus \{\xi\}$ be a regular loop with

$$\|\sigma - \sigma_n\|_{\sup} < \frac{1}{n}.$$

By Lemma A.2 there is a sequence of pairs $\{(t_n, s_n)\}$, $t_n < s_n$, $t_n, s_n \in [0,1]$, $n \geq n_0$ such that $\operatorname{rot}_\xi \sigma_{n|[t_n,s_n]} = 1$. Going to subsequences if necessary, one has $\bar{t} = \lim_{n\to\infty} t_n$ and $\bar{s} = \lim_{n\to\infty} s_n$. It is an elementary exercise to show that $\bar{t} < \bar{s}$, $\sigma(\bar{t}) = \sigma(\bar{s})$ and $\operatorname{rot}_\xi \sigma_{|[\bar{t},\bar{s}]} = 1$. □

Let $I = [a,\xi] \subset l$ and let $j_*\colon \pi_1(\mathbb{R}^2 \setminus I) \to \pi_1(\mathbb{R}^2 \setminus \{a,\xi\})$ be a map induced by the natural inclusion.

Proposition A.3. *If* $\sigma\colon [0,1] \to \mathbb{R}^2 \setminus \{a,\xi\}$ *is a loop satisfying* $[q] \in im j_*$ *and* $\operatorname{rot}_\xi q \geq 2$, *then there exist* $t, s \in [0,1]$ *such that* $t < s$, $q(t) = q(s)$ *and* $\operatorname{rot}_a(q_{|[t,s]}) = \operatorname{rot}_\xi(q_{|[t,s]}) = 1$.

The proof is omitted.

Remark A.4. Similar results can be proven if one replaces a loop $\sigma\colon [0,1] \to \mathbb{R}^2 \setminus \{\xi\}$ by $q\colon \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$ with $\lim_{t\to\pm\infty} q(t) = p \in \mathbb{R}^2 \setminus \{a,\xi\}$.

Acknowledgment

This research was supported by the National Science Centre of Poland, grant no. 2011/03/B/ST1/04533.

References

[1] M. L. Bertotti and L. Jeanjean, *Multiplicity of homoclinic solutions for singular second-order conservative systems*. Proc. Roy. Soc. Edinburgh Sect. A **126** (1996), 1169–1180.

[2] S. V. Bolotin, *Variational criteria for nonintegrability and chaos in Hamiltonian systems*. In: Hamiltonian Mechanics (Toruń, 1993), NATO Adv. Sci. Inst. Ser. B Phys. 331, Plenum, New York, 1994, 173–179.

[3] S. V. Bolotin and P. Negrini, *A variational criterion for nonintegrability*. Russ. J. Math. Phys. **5** (1997), 415–436.

[4] M. J. Borges, *Heteroclinic and homoclinic solutions for a singular Hamiltonian system*. European J. Appl. Math. **17** (2006), 1–32.

[5] P. Caldiroli and L. Jeanjean, *Homoclinics and heteroclinics for a class of conservative singular Hamiltonian systems*. J. Differential Equations **136** (1997), 76–114.

[6] P. Caldiroli and M. Nolasco, *Multiple homoclinic solutions for a class of autonomous singular systems in* $\mathbb{R}^2$. Ann. Inst. H. Poincaré Anal. Non Linéaire **15** (1998), 113–125.

[7] W. B. Gordon, *Conservative dynamical systems involving strong forces*. Trans. Amer. Math. Soc. **204** (1975), 113–135.

[8] M. Izydorek and J. Janczewska, *Heteroclinic solutions for a class of the second order Hamiltonian systems.* J. Differential Equations **238** (2007), 381–393.

[9] M. Izydorek and J. Janczewska, *The shadowing chain lemma for singular Hamiltonian systems involving strong forces.* Cent. Eur. J. Math. **10** (2012), 1928–1939.

[10] M. Izydorek and J. Janczewska, *Connecting orbits for a periodically forced singular planar Newtonian system.* J. Fixed Point Theory Appl. **12** (2012), 59–67.

[11] J. Janczewska, *The existence and multiplicity of heteroclinic and homoclinic orbits for a class of singular Hamiltonian systems in* $\mathbb{R}^2$. Boll. Unione Mat. Ital. (9) **3** (2010), 471–491.

[12] J. Janczewska and J. Maksymiuk, *Homoclinic orbits for a class of singular second order Hamiltonian systems in* $\mathbb{R}^3$. Cent. Eur. J. Math. **10** (2012), 1920–1927.

[13] P. H. Rabinowitz, *Periodic and heteroclinic orbits for a periodic Hamiltonian system.* Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), 331–346.

[14] P. H. Rabinowitz, *Homoclinics for an almost periodically forced singular Hamiltonian system.* Topol. Methods Nonlinear Anal. **6** (1995), 49–66.

[15] P. H. Rabinowitz, *Multibump solutions for an almost periodically forced singular Hamiltonian system.* Electron. J. Differential Equations **1995** (1995), 1–21.

[16] P. H. Rabinowitz, *Homoclinics for a singular Hamiltonian system.* In: Geometric Analysis and The Calculus of Variations (J. Jost, ed.), Int. Press, Cambridge, MA, 1996, 267–296.

[17] P. H. Rabinowitz, *A multibump construction in a degenerate setting.* Calc. Var. Partial Differential Equations **5** (1997), 159–182.

[18] L. P. Shil'nikov, *Homoclinic trajectories: From Poincaré to the present.* In: Mathematical Events of the Twentieth Century, Springer, Berlin, 2006, 347–370.

[19] K. Tanaka, *Homoclinic orbits for a singular second order Hamiltonian system.* Ann. Inst. H. Poincaré Anal. Non Linéaire **7** (1990), 427–438.

Marek Izydorek
Faculty of Applied Physics and Mathematics
Gdańsk University of Technology
Narutowicza 11/12
80-233 Gdańsk
Poland
e-mail: `izydorek@mif.pg.gda.pl`

Joanna Janczewska
Faculty of Applied Physics and Mathematics
Gdańsk University of Technology
Narutowicza 11/12
80-233 Gdańsk
Poland
e-mail: `janczewska@mif.pg.gda.pl`