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Wave dispersion relations in peridynamics: Influence of kernels and similarities to nonlocal elasticity theories

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ABSTRACT

We investigate the wave dispersion relations of an infinite elastic bar within the framework of linear bond-based peridynamics. This nonlocal integral-type model accounts for long-range interactions, which become significant at small scales and in cases of damage and fracture. Since a key element of this material model is the kernel function, we derive dispersion curves for several kernel choices. Notably, for non-singular kernels, we observe negative group velocities, indicating that peridynamics can describe materials with anomalous dispersion. By comparing one-dimensional (1D) peridynamics with the 1D nonlocal elasticity of Eringen's type, we highlight similarities between the two models in terms of dispersion behavior.

1. Introduction

Recently, there has been a significant increase in interest in nonlocal models of continua and structures. These models have found various applications in modeling long-range interactions, which are essential at small scales (see, for example, the discussions in de Gennes (1981), Kröner (1967), Maugin (2017)). They are also particularly relevant in the study of damage and fracture, as exemplified by the nonlocal elasticity theory proposed by Eringen (1972), Eringen and Edelen (1972) and summarized in Eringen (2002). The integral constitutive relations for stresses introduced in these works can, in some cases, be transformed into a differential form, commonly referred to as stress-gradient elasticity–a complementary counterpart to strain-gradient elasticity (Maugin, 2017). This approach, sometimes with modifications, has been extensively studied in the literature (see, for example, Aifantis (1999), Askes and Aifantis (2011), Barretta, Caporale, Luciano, and Vaccaro (2023), Barretta, Luciano, de Sciarra, and Vaccaro (2024), Khodabakhshi and Reddy (2015), Penna (2023) and the references therein).

In addition to Eringen-type nonlocal elasticity, another nonlocal model, known as peridynamics, has been proposed by Silling (2000) (see also Madenci and Oterkus (2013), Silling and Lehoucq (2010)). Peridynamics has also found applications in modeling long-range interactions at small scales and in describing damage and fracture (see, for example, Chan and Chen (2023), Izadi, Das, Fantuzzi, and Trovalusci (2024), Naumenko, Pander, and Würkner (2022) and the references therein). Wave phenomena, including dispersion relations, have been analyzed in several studies (Alebrahim, Packo, Zaccariotto, & Galvanetto, 2022; Bažant, Luo, Chau, & Bessa, 2016; Chan & Chen, 2021; Chen, Peng, Jafarzadeh, & Bobaru, 2023; Gu, Zhang, Huang, & Yv, 2016; Mutnuri & Gopalakrishnan, 2018; Silling, Zimmermann, & Abeyaratne, 2003; Wang & Huang, 2019; Weckner & Abeyaratne, 2005; Yang, Ma, Oterkus, Oterkus, & Naumenko, 2023). In the case of state-based peridynamics, dispersion has been discussed in Alebrahim

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(2023), Butt, Timothy, and Meschke (2017). These studies have shown that dispersion behavior strongly depends on the choice of kernel functions. While previous works have typically considered one or two kernels, a comparative analysis of five kernels in the static case was presented by Naumenko, Yang, Ma, and Chen (2023). Dispersion properties play a crucial role in wave propagation and the dynamic response of materials in general (see Achenbach (1973), Whitham (1999)). Moreover, experimental data and/or theoretical estimations of dispersion can be used both for validating nonlocal models and for determining material parameters.

The aim of this paper is to analyze dispersion relations within the one-dimensional (1D) bond-based peridynamics framework, considering different kernel functions. It is worth noting that the 1D model is not only of theoretical interest but also has practical applications, for example in modeling brittle fibers such as glass fibers, which are widely used in modern engineering.

The remainder of this paper is organized as follows. In Section 2, starting from the classical case, we introduce the basic equation of motion and the dispersion relations for peridynamics. In Section 3, we derive the dispersion relations–i.e., the relations between frequency and wavenumber–for several kernel functions and present the corresponding dispersion curves. Finally, in Section 4, we compare the dispersion behavior in peridynamics with that in nonlocal elasticity.

2. Equations of motion

In the following we consider infinitesimal deformations of an infinite homogeneous elastic bar. So we introduce the 1D displacement field u as a smooth enough function of a spatial coordinate x and time t:

$$u = u(x,t).$$

2.1. Simple bar

Within the linear elasticity we have Hooke's law

$$\sigma = E\varepsilon,$$

where σ is the stress, E is the Young modulus, and $\varepsilon \equiv u'$ is the strain. As a result, we get the equation of motion in the form of the wave equation (Achenbach, 1973)

$$\rho\ddot{u} = Eu'', \tag{1}$$

where ρ is the mass density. Hereinafter the prime stands for the derivative with respect to x, whereas the overdot denotes the derivative with respect to t.

Looking for a solution of (1) in the harmonic form

$$u = U\exp\left[i(kx - \omega t)\right], \tag{2}$$

where U is an amplitude, $i = \sqrt{-1}$ is the imaginary unit, k and ω are the wavenumber and frequency, respectively, we get the dispersion relation

$$\omega = c_\infty k, \quad c_\infty = \sqrt{\frac{E}{\rho}}, \tag{3}$$

where c_∞ is the speed of the longitudinal waves. Obviously, Eq. (1) describes non-dispersive waves.

Motivated by experimental data and comparisons with solutions obtained from 3D linear elasticity, various extensions of (1) related to dispersive waves can be found in the literature. These include Rayleigh's and Bishop's equations (Bishop, 1952; Strutt, 1945), the 3D Pochhammer–Chree solution presented in Royer and Dieulesaint (2000), and the reviews in Artobolevskiy, Bobrovmitskiy, and Genkin (1979), Eremeyev (2025).

2.2. Bond-based peridynamics

Following Silling (2000), Silling and Lehoucq (2010) let us now consider 1D wave propagation within the linear bond-based peridynamics. Instead of (1) we have the integro-differential equation

$$\rho\ddot{u} = \frac{E}{\varkappa(\delta)}\int_{-\delta}^{\delta} F(\xi)\left[u(x+\xi,t) - u(x,t)\right]\, d\xi, \tag{4}$$

where δ is the horizon size, $F(\xi)$ is an even kernel function such that

$$F(-\xi) = F(\xi), \quad \lim_{\xi\to 0} \xi^2 F(\xi) < 0,$$

and

$$\varkappa(\delta) = \frac{1}{2}\int_{-\delta}^{\delta} \xi^2 F(\xi)\, d\xi.$$

For $\delta \to 0$ and some reasonable assumptions about $F(\xi)$ Eq. (4) could be transformed to the classical one.

Although the concept of stress does not play an essential role in peridynamics, see, e.g., comments by Madenci and Oterkus (2013), Silling and Lehoucq (2008, 2010), Silling et al. (2003), Weckner and Abeyaratne (2005). Just to have a possibility to make a connection between this model and other local and non-local theories, the effective stress could be introduced similar to Silling et al. (2003), Weckner and Abeyaratne (2005) as follows

$$\sigma(x,t) = \frac{E}{\varkappa(\delta)} \int_0^{\delta} \int_0^{\delta-\xi} F(\xi+\eta)\,[u(x+\eta,t) - u(x-\xi,t)]\; d\eta\, d\xi, \tag{5}$$

In particular, for homogeneous deformations of the form $u(x,t) = x\varepsilon(t)$ we came to Hooke's law.

Substituting (2) into Eq. (4) we came to

$$\omega^2 = c_\infty^2 C(k), \quad C(k) = \frac{1}{\varkappa} \int_{-\delta}^{\delta} F(\xi)[1-\cos k\xi]\, d\xi. \tag{6}$$

As a result, we get the dispersion relation in the form

$$\omega = c_\infty A(k), \quad A(k) = [C(k)]^{1/2}, \tag{7}$$

see e.g. Silling et al. (2003), Weckner and Abeyaratne (2005) for more details.

Using (7) we obtain the phase c and group v velocities given by

$$c \equiv \frac{\omega}{k} = c_\infty \frac{A(k)}{k}, \quad v \equiv \frac{d\omega}{dk} = c_\infty \frac{dA(k)}{dk}. \tag{8}$$

3. Kernel functions and dispersion relations

The kernel function $F(\xi)$ determines the material response within the bond-based peridynamics. Some useful kernel functions can be found in the literature and are summarized below as in Naumenko et al. (2023)

$$F_1(\xi) = \begin{cases} 1, & |\xi| \le \delta, \\ 0 & |\xi| > \delta \end{cases}, \tag{9}$$

$$F_2(\xi) = \begin{cases} 1-|\xi|, & |\xi| \le \delta, \\ 0 & |\xi| > \delta \end{cases}, \tag{10}$$

$$F_3(\xi) = \begin{cases} \dfrac{1}{|\xi|}, & |\xi| \le \delta, \\ 0 & |\xi| > \delta \end{cases}, \tag{11}$$

$$F_4(\xi) = \begin{cases} \exp\left(-4\dfrac{\xi^2}{\delta^2}\right), & |\xi| \le \delta, \\ 0 & |\xi| > \delta \end{cases}. \tag{12}$$

In particular, the uniform kernel function (9) was used by Bažant et al. (2016), Silling et al. (2003), Yang et al. (2023). The exponential kernel (12) was also used by Weckner and Abeyaratne (2005) along with three other kernels, where the corresponding dispersion curves were obtained for an infinite horizon size. In the dispersion analysis conducted by Mikata (2012, 2019), several kernel functions were considered, including the uniform (9), triangular (10), and exponential (12) kernels. A static analysis involving the hyperbolic kernel (11) was provided by Yang et al. (2023).

Some other kernel functions could be proposed such as the following

$$F_5(\xi) = \begin{cases} \left(\delta^2-\xi^2\right)^m, & |\xi| \le \delta, \\ 0 & |\xi| > \delta \end{cases}, \tag{13}$$

$$F_6(\xi) = \begin{cases} \sin^m \dfrac{\pi\xi}{\delta}, & |\xi| \le \delta, \\ 0 & |\xi| > \delta \end{cases}, \tag{14}$$

$$F_7(\xi) = \begin{cases} \dfrac{1}{|\xi|^2}, & |\xi| \le \delta, \\ 0 & |\xi| > \delta \end{cases}. \tag{15}$$

where m is an even number. Note that the kernel functions F_3 and F_5 are singular. These functions tend to infinity at $\xi \to 0$.

The introduced kernels lead to the following values of $\varkappa$ and functions

$$D(k) \equiv \int_0^{\delta} F(\xi)[1-\cos k\xi]\, d\xi = \frac{1}{2}\varkappa C(k)$$

used in the dispersion Eqs. (6) and (7)

$$\varkappa_1 = \frac{1}{3}\delta^3, \qquad D_1(k) = \delta - \frac{\sin(k\delta)}{k},$$

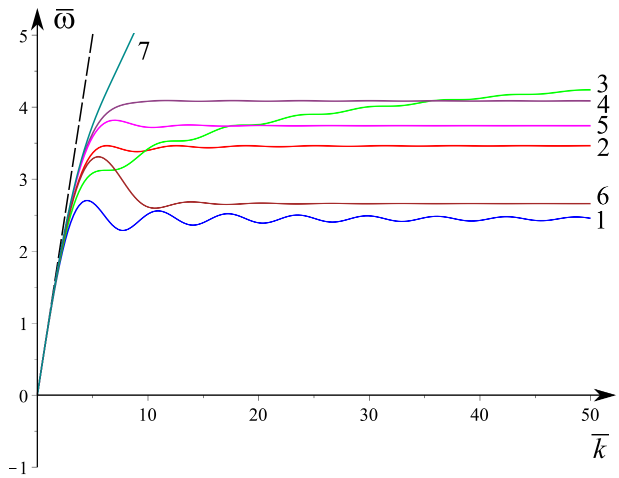

Fig. 1. Normalized frequency $\overline{\omega} = \frac{\delta\omega}{c_\infty}$ versus normalized wave number $\overline{k} = k\delta$. Dashed line corresponds to (3). Curves 1, 2, …, 7 relates to kernels F_1, F_2, …, F_7, respectively.

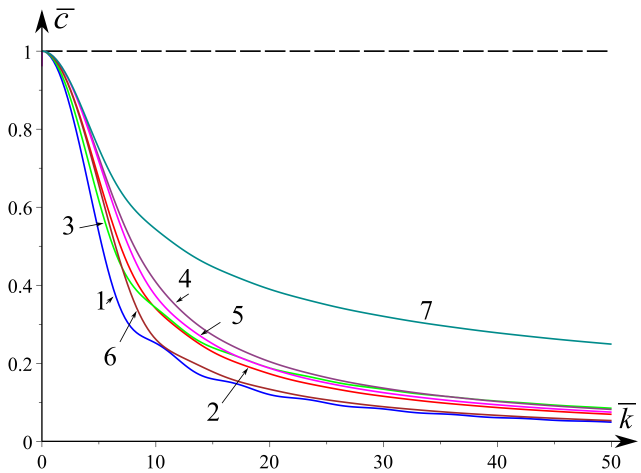

Fig. 2. Normalized phase velocity $\overline{c} = \frac{c}{c_\infty}$ versus normalized wave number $\overline{k} = k\delta$. Dashed line corresponds to the equation $c = c_\infty$. Curves 1, 2, …, 7 relates to kernels F_1, F_2, …, F_7, respectively.

$$\varkappa_2 = \frac{1}{3}\delta^3 - \frac{1}{4}\delta^4, \qquad D_2(k) = \frac{1}{2} + \frac{\cos(k) - 1}{k^2},$$

$$\varkappa_3 = \frac{1}{2}\delta^2, \qquad D_3(k) = \gamma + \ln(k\delta) - \mathrm{Ci}(k\delta),$$

$$\varkappa_4 = \frac{\delta^3}{32}\left(\sqrt{\pi}\,\mathrm{erf}(2) - \frac{4}{\mathrm{e}^4}\right), \qquad D_4(k),$$

$$\varkappa_5 = \frac{1}{3}\delta^4 - \frac{2}{5}\delta^2 + \frac{1}{7}, \qquad D_5(k) = \frac{8}{15}\delta^5 + \frac{\sin(k\delta)\,\delta^2 k^2 + 3\,k\cos(k\delta)\,\delta - 3\,\sin(k\delta)}{k^5},$$

$$\varkappa_6 = \frac{1}{12}\,\frac{\delta^3\left(2\pi^2 - 3\right)}{\pi^2}, \qquad D_6(k) = \frac{4\pi^2 k\delta - \delta^3 k^3 - 4\pi^2\sin(k\delta)}{2k\left(-k^2\delta^2 + 4\pi^2\right)},$$

$$\varkappa_7 = \delta, \qquad D_7(k) = \frac{\mathrm{Si}(k\delta)\,k\delta + \cos(k\delta) - 1}{\delta},$$

where $\gamma \approx 0.577721$ is Euler's constant, Ci and Si is the cosine and sine integral, see Abramowitz and Stegun (1972). We have used the value $m = 2$, and we omit the awkward expression for $D_4(k)$.

The dispersion curves are shown in Fig. 1. Here $\overline{\omega} = \frac{\delta\omega}{c_\infty}$, $\overline{k} = k\delta$. The dashed line corresponds to the classical elasticity (3), while the ith curve corresponds to the F_i kernel, $i = 1, \ldots, 7$. At $k = 0$ all curves have the same tangent given by (3).

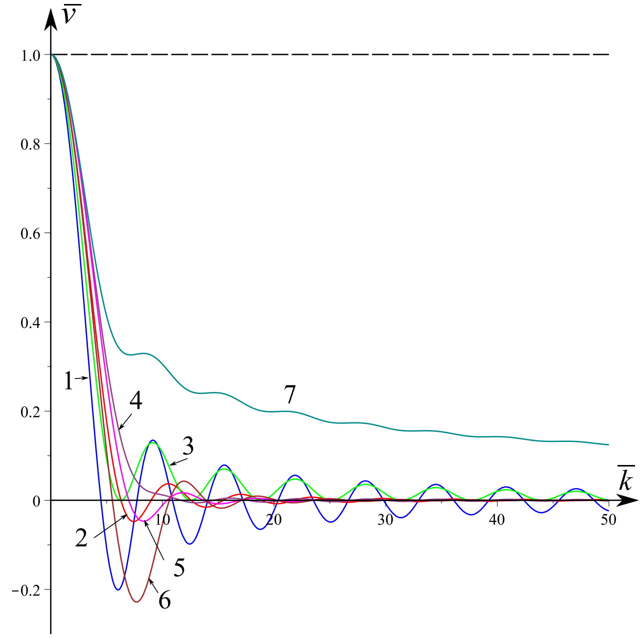

Fig. 3. Normalized group velocity $\overline{v} = \frac{v}{c_\infty}$ versus normalized wave number $\overline{k} = k\delta$. Dashed line corresponds to the equation $c = c_\infty$. Curves 1, 2, …, 7 relates to kernels F_1, F_2, …, F_7, respectively.

In Fig. 2, we show the normalized phase velocities, $\overline{c} = \frac{c}{c_\infty}$. Clearly, for all kernel functions, $c_i < c_\infty$, meaning that waves propagate at lower velocities compared to classical elasticity. At $k = 0$, all curves have the same horizontal tangent, $c = c_\infty$.

Finally, in Fig. 3, we present the normalized group velocities, $\overline{v} = \frac{v}{c_\infty}$. Except for singular kernels, all other curves intersect the horizontal axis, meaning that for certain wavelength ranges, the group velocity becomes negative. This phenomenon is associated with so-called anomalous dispersion. For hyperbolic kernels, the group velocity remains non-negative but reaches zero at certain values of k, similar to other curves for non-singular kernels. In this case, waves with the corresponding wavelengths do not transmit energy.

4. Comparison with the nonlocal elasticity

Let us compare the dispersion relations with those derived within nonlocal elasticity. Following Eringen (2002), Eringen and Edelen (1972), in the 1D case the constitutive relation has the form

$$\sigma = E \int_{-\infty}^{\infty} K(x-\xi)\,\varepsilon(\xi,t)\,d\xi, \tag{16}$$

where E is again Young's modulus, K is an even kernel function normalized as follows

$$\int_{-\infty}^{\infty} K(\xi)\,d\xi = 1.$$

Now the equation of motion is given by

$$\rho\ddot{u} = \sigma' \equiv E\left[\int_{-\infty}^{\infty} K(x-\xi)\,\varepsilon(\xi,t)\,d\xi\right]'. \tag{17}$$

Looking for a solution of (17) in the harmonic form we came to the equation

$$\omega^2 = c_\infty^2 k^2 \overline{K}(k), \quad \overline{K}(k) = \int_{-\infty}^{\infty} K(\eta)e^{-ik\eta}\,d\eta = \int_{-\infty}^{\infty} K(\eta)\cos k\eta\,d\eta. \tag{18}$$

This gives us the dispersion relation

$$\omega \equiv \omega_{NL}(k) = c_\infty B(k), \quad B(k) = k\left[\overline{K}(k)\right]^{1/2}. \tag{19}$$

So the phase velocity became

$$c_{NL} = c_\infty \frac{B(k)}{k} = c_\infty\left[\overline{K}(k)\right]^{1/2}. \tag{20}$$

Note that if $c_{NL}(k) \in L^2(\mathbb{R})$ we can find the kernel function using (20) and the inverse Fourier transform by the formula

$$K(\eta) = \frac{1}{2\pi c_\infty^2} \int_{-\infty}^{\infty} c_{NL}^2(k) e^{ik\eta}\, dk. \tag{21}$$

Askes and Aifantis (2011) provided a detailed analysis of the dispersion relations within stress and strain gradient elasticity, where ω^2 takes the form of a rational function of k.

Introducing non-local model Eringen was motivated by the comparison of dispersion with the lattice dynamics, see Eringen (1972, 2002), Eringen and Edelen (1972). In particular, a kernel can be taken to fit the dispersion curves obtained within the lattice dynamics.

A similar idea can be applied to peridynamics. Moreover, discussing the correspondence between the equation and the dispersion relation (Whitham, 1999, pp. 367, 368) mentioned that the equation can be constructed to give any desired phase velocity and consequently any desired dispersion function.

Comparing dispersion relations (7) and (19) we can see that they coincide to each other when

$$A(k) = B(k), \tag{22}$$

or if

$$\frac{1}{\varkappa} \int_{-\delta}^{\delta} F(\xi)[1 - \cos k\xi]\, d\xi = k^2 \int_{-\infty}^{\infty} K(\eta) \cos k\eta\, d\eta. \tag{23}$$

For example, in the case of the uniform kernel function (9) Eq. (23) transforms to the determination of an inverse Fourier transform of the function

$$\overline{K}(k) = \frac{6}{\delta^2 k^2} \left[1 - \frac{\sin k\delta}{k\delta}\right].$$

This correspondence means that the same dispersion could be described using both non-local theories. Of course, this conclusion is based on the analysis of the 1D case of an infinite bar, i.e. we have not considered here spatial effects and boundary conditions. We have also assumed that all mathematical operations are meaningful.

Conclusions

Within the 1D linear bond-based peridynamics, we have obtained dispersion relations for several kernel functions and plotted the corresponding phase and group velocity curves. Examination of the group velocity shows that it can be negative, which is related to the so-called anomalous dispersion observed in some metamaterials, see e.g. Bigoni, Guenneau, Movchan, and Brun (2013), Bossart and Fleury (2023), Park and Oh (2019) and the references therein.

Notably, all considered non-singular kernels produce qualitatively similar oscillatory dispersion curves for short-wavelength waves ($k\delta \geq 5$), with a horizontal asymptote. In contrast, for long-wavelength waves ($k\delta \leq 2$), all curves nearly coincide.

As already mentioned by Mikata (2012), peridynamics could be useful for modeling materials with anomalous dispersion. Comparing peridynamics with nonlocal elasticity, we emphasize that both theories can exhibit the same dispersion behavior. In particular, depending on the chosen kernel, nonlocal elasticity may also describe anomalous dispersion.

CRediT authorship contribution statement

Victor A. Eremeyev: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Project administration, Funding acquisition, Formal analysis, Conceptualization. **Konstantin Naumenko:** Writing – review & editing, Writing – original draft, Visualization, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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