This is a post-peer-review, pre-copyedit version of an article published in **Lobachevskii Journal of Mathematics**. The final authenticated version is available online at: http://dx.doi.org/10.1134/S1995080220100078

Postprint of: Eremeyev, V.A., dell'Isola, F., Weak Solutions within the Gradient-Incomplete Strain-Gradient Elasticity, Lobachevskii J. Math, Vol. 41, No 10 (2020), pp. 1992–1998, DOI: 10.1134/S1995080220100078

Weak Solutions within the Gradient-Incomplete Strain-Gradient Elasticity

V. A. Eremeyev[1, 2, *] and **F. dell'Isola**[3, 2, **]

[1] *Faculty of Civil and Environmental Engineering, Gdańsk University of Technology, ul. Gabriela Narutowicza 11/12, 80-233 Gdańsk, Poland*
[2] *Research Institute for Mechanics, National Research Lobachevsky State University of Nizhni Novgorod, Gagarin Avenue, 23, 603950 Nizhni Novgorod, Russia*
[3] *International Research Center on Mathematics and Mechanics of Complex System (M&MOCS), Università dell'Aquila, l'Aquila, Italy*

Abstract—In this paper we consider existence and uniqueness of the three-dimensional static boundary-value problems in the framework of so-called gradient-incomplete strain-gradient elasticity. We call the strain-gradient elasticity model gradient-incomplete such model where the considered strain energy density depends on displacements and only on some specific partial derivatives of displacements of first- and second-order. Such models appear as a result of homogenization of pantographic beam lattices and in some physical models. Using anisotropic Sobolev spaces we analyze the mathematical properties of weak solutions. Null-energy solutions are discussed.

2010 Mathematical Subject Classification: 74E30, 35J58, 35H10, 46E35, 46E35

Keywords and phrases: *strain-gradient elasticity, weak solution, existence, uniqueness, gradient incomplete strain-gradient elasticity.*

1. INTRODUCTION

Recent progress in technologies including additive ones results in appearance of new classes of materials called metamaterials or architectured materials. Among these it is worth to mention so-called pantographic beam lattices discussed in [11, 12]. Such material consists of few families of long elastic beams connected through relatively small and soft pivots. The related continual description of such materials results in a specific class of strain-gradient materials that is materials with a strain energy density dependent on first and second gradients of displacements. The general framework of the strain-gradient elasticity was established in the seminal papers by Toupin and Mindlin [26, 27, 33]. It is worth also to mention that an origin of such models can be found in earlier works of the continuum mechanics founders, see [3, 9, 10, 23, 24].

Following the Toupin—Mindlin approach a strain energy density has the form

$$\mathcal{W} = \mathcal{W}(\mathbf{e}, \mathbf{k}), \tag{1}$$

where

$$\mathbf{e} = \tfrac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \quad \mathbf{k} = \nabla\nabla\mathbf{u}$$

is the strain tensor and the second gradient of $\mathbf{u}$, $\mathbf{u} = (u_1, u_2, u_3)$ is the vector of displacements, and ∇ is the three-dimensional (3D) nabla-operator [15, 32].

In the framework of infinitesimal deformations $\mathcal{W}$ is a quadratic form of $\mathbf{e}$ and $\mathbf{k}$

$$\mathcal{W} = \tfrac{1}{2}\mathbf{e} : \mathbf{C} : \mathbf{e} + \tfrac{1}{2}\mathbf{k} \,\vdots\, \mathbf{D} \,\vdots\, \mathbf{k}, \tag{2}$$

* E-mail: eremeyev.victor@gmail.com
** E-mail: francescodellisola@uniroma1.it

where $\mathbf{C}$ and $\mathbf{D}$ are forth- and six-order tensors of elastic moduli, respectively. Here ":" and "$\vdots$" stand for scalar (inner) products in the spaces of second- and third-order tensors, respectively. The standard assumption is that $\mathbf{C}$ and $\mathbf{D}$ to be positive definite

$$\mathbf{e} : \mathbf{C} : \mathbf{e} \geq C_1\, \mathbf{e} : \mathbf{e}, \quad \mathbf{k} \vdots \mathbf{D} \vdots \mathbf{k} \geq C_2\, \mathbf{k} \vdots \mathbf{k} \tag{3}$$

with $C_1 > 0$, $C_2 > 0$. This assumption leads to the existence and uniqueness of weak solutions of the corresponding boundary-value problems [20, 22].

The homogenization of pantographic lattices leads to a strain energy density given by (2) but without positive definiteness. Here we have only

$$\mathbf{k} \vdots \mathbf{D} \vdots \mathbf{k} \geq 0. \tag{4}$$

As a result, the standard definition of coercitivity is violated and should be modified properly. In order to prove the existence and uniqueness of weak solutions for in-plane and out-of-plane deformations of pantographic sheets in [16, 17] the anisotropic Sobolev spaces were used. This class of functional spaces was introduced by Sergey Nikolskii [28], see also [4, 5, 34].

The aim of this paper is to extend the results of [16, 17] to three-dimensional pantographic beam lattices with perfect pivots which undergo infinitesimal deformations. The paper is organized as follows. First, we introduce a strain energy density in Section 2. In Section 3 we discuss null-energy solutions. This class of deformations includes the rigid-body deformations and should be avoided in order to get the uniqueness of solutions. Finally, in Section 4 we formulate and prove the theorems of uniqueness and existence of weak solutions. Here we use the anisotropic Sobolev spaces as an energy functional space.

2. STRAIN ENERGY DENSITY

Following [6, 16, 17, 30, 31] let us briefly introduce the strain energy density of a three-dimensional pantographic lattice. It consists of three orthogonal families of elastic beams connected through small soft pivots, see Fig. 1. In what follows we neglect bending and torsional deformations of the pivots. Such pivots we call perfect ones. In addition we neglect torsional deformations of beams.

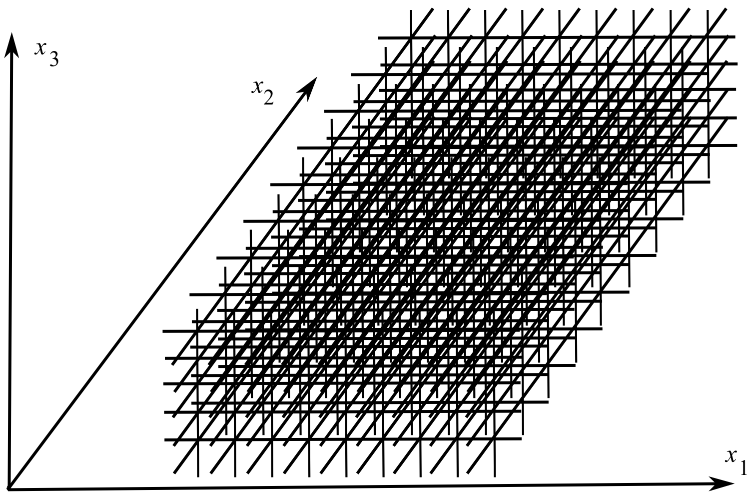

Figure 1. Three-dimensional pantographic beam lattice.

Using an approach similar to [17] we come to a strain energy density in the following form

$$\begin{aligned} \mathcal{W} =& \frac{\mathbb{K}_e^{(1)}}{2} u_{1,1}^2 + \frac{\mathbb{K}_e^{(2)}}{2} u_{2,2}^2 + \frac{\mathbb{K}_e^{(3)}}{2} u_{3,3}^2 \\ &+ \frac{\mathbb{K}_b^{(1)}}{2} \left(u_{2,11}^2 + u_{3,11}^2\right) + \frac{\mathbb{K}_b^{(2)}}{2} \left(u_{1,22}^2 + u_{3,22}^2\right) + \frac{\mathbb{K}_b^{(3)}}{2} \left(u_{1,33}^2 + u_{2,33}^2\right), \end{aligned} \tag{5}$$

where the stiffness parameters $\mathbb{K}_e^{(i)} > 0$ and $\mathbb{K}_b^{(i)} > 0$, u_1, u_2, u_3 are Cartesian components of $\mathbf{u}$, $i = 1, 2, 3$, are related to the extensional and bending stiffnesses of the beams at the interpivot scale, respectively. Hereinafter indices after comma denote derivatives with respect to x_i, $i = 1, 2, 3$, where x_i are Cartesian coordinates. So, for example, $u_{1,1}$ is the partial derivative of u_1 with respect to x_1, $u_{1,22} = \frac{\partial^2 u_1}{\partial x_2^2}$, etc. Eq. (5) is a straightforward 3D generalization of 2D models of pantographic sheets [17].

Let us underline some similarities of (5) with other physical models known in the literature. Eq. (5) is a 3D analog of a strain energy of an extensible beams undergoing in-plane deformations

$$\mathcal{W} = \tfrac{1}{2}\mathbb{K}_e(u_1')^2 + \tfrac{1}{2}\mathbb{K}_b(u_2'')^2, \tag{6}$$

where $\mathbb{K}_e > 0$ and $\mathbb{K}_b > 0$ are the extensional and bending stiffnesses, $u_1 = u_1(x)$ and $u_2 = u_2(x)$ are longitudinal and transverse displacements, respectively, x is the coordinate along the beam axis, and the prime stands for derivative with respect to x.

Another example is the strain energy of small deformations of the smectics A given by [7, 19, 29]

$$\mathcal{W} = \tfrac{1}{2}\mathbb{B}(u_{,3})^2 + \tfrac{1}{2}\mathbb{K}\left(u_{,11} + u_{,22}\right)^2, \tag{7}$$

where $\mathbb{B}$ and $\mathbb{K}$ are elastic moduli describing the longitudinal stiffness along x_3-direction and the bending stiffness of the smectic layers, respectively. Here the displacement vector has the simple form $\mathbf{u} = (0, 0, u(x_1, x_2, x_3))$.

Obviously, Eqs. (6) and (7) give 1D and 3D scalar examples of (5). Other physical models with non-symmetric appearance of second-order derivatives are discussed in [13]. We call the model based on (5) gradient-incomplete as $\mathcal{W}$ does not contain all second-order derivatives. Indeed, for example, $\mathcal{W}$ contains only $u_{1,1}$, $u_{1,22}$, and $u_{1,22}$ while $u_{1,11}$ and mixed derivatives are absent.

3. NULL-ENERGY SOLUTIONS

Let us recall that for classic linear elasticity there is an energy null-space which reduces to the so-called infinitesimal rigid-body deformations. The energy null-space is a set of admissible functions for which the given strain energy density is vanishing. The analysis of these null-energy solutions is important element of the analysis of existence and uniqueness of weak solutions, see [8, 14, 18]. Let us find null-energy solutions considering solutions of

$$\mathcal{W} = 0. \tag{8}$$

In classic linear elasticity it is known that such solutions are so-called rigid-body deformations given by [14]

$$\mathbf{u} = \mathbf{a} + \mathbf{b} \times \mathbf{x}, \tag{9}$$

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, $\mathbf{x} = (x_1, x_2, x_3)$ is the position vector, and $\times$ is the cross-product. So in the linear elasticity the null-energy solutions constitute a six-dimensional subspace.

For the constitutive equation (5) we also have finite-dimensional null-space. Indeed, considering (8) we get the solutions

$$u_1 = a_1 yz + b_1 z + c_1 z + d_1, \tag{10}$$

$$u_2 = a_2 xz + b_1 x + c_2 z + d_2, \tag{11}$$

$$u_3 = a_3 xy + b_3 x + c_3 y + d_3, \tag{12}$$

where a_i, b_i, c_i, and d_i, $i = 1, 2, 3$, are constants. So for the considered model we again have finite-dimensional null-energy subspace. Unlike (9) here we have 12 additional parameters so the null-space is 12-dimensional. Obviously, (10)–(12) includes rigid-body deformations (9). Thus, the gradient-incompleteness of (5) results in a number of solutions with zero energy.

4. EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS

Let us consider an elastic solid body B with the strain energy density (5). B occupies a bounded volume V with smooth enough boundary $S = \partial V$, see, e.g., [2, 21, 25] for the detailed requirements to S. Equilibrium conditions of B follow from the virtual principle

$$\delta \int_V \mathcal{W}\, dV - \delta A = 0, \tag{13}$$

where δA is a work of external loads, see [1, 3] and references therein for virtual work principle in the framework of the higher gradient elasticity models.

For simplicity we consider δA as in the case of linear elasticity [8, 14, 18]

$$\delta A = \int_V \mathbf{f} \cdot \delta \mathbf{u}\, dV + \int_{S_1} \mathbf{t} \cdot \delta \mathbf{u}\, dS, \tag{14}$$

where $\mathbf{f}$ and $\mathbf{t}$ are given vector functions of body and surface forces, respectively, "$\cdot$" stands for scalar product of vectors, and $S_1 \subset S$ is a part of S where a traction $\mathbf{t}$ is given.

In the following we consider dimensionless form of $\mathcal{W}$ and other introduced quantities. Moreover, without loss of generality we use the following dimensionless form of $\mathcal{W}$

$$2\mathcal{W} = u_{1,1}^2 + u_{2,2}^2 + u_{3,3}^2 + u_{2,11}^2 + u_{3,11}^2 + u_{1,22}^2 + u_{3,22}^2 + u_{1,33}^2 + u_{2,33}^2. \tag{15}$$

Let us consider the quadratic functional $N(\mathbf{u})$ given by

$$N(\mathbf{u}) = \int_V 2\mathcal{W}\, dV. \tag{16}$$

Obviously, $N(\mathbf{u})$ constitutes a seminorm, as $N(\mathbf{u}) = 0$ results in (10)–(12).

Let us recall the definitions of norms and seminorms in the anisotropic Sobolev spaces $H_1 = H_2^{(1,2,2)}$, $H_2 = H_2^{(2,1,2)}$, and $H_3 = H_2^{(2,2,1)}$. Here we have [28, 34]

$$\begin{aligned}
\|u\|_1^2 \equiv \|u\|^2_{H_2^{(1,2,2)}} &= \int_V \left(u^2 + u_{,1}^2 + u_{,22}^2 + u_{,33}^2\right) dV,\\
\|u\|_2^2 \equiv \|u\|^2_{H_2^{(2,1,2)}} &= \int_V \left(u^2 + u_{,2}^2 + u_{,11}^2 + u_{,33}^2\right) dV,\\
\|u\|_3^2 \equiv \|u\|^2_{H_2^{(2,2,1)}} &= \int_V \left(u^2 + u_{,3}^2 + u_{,22}^2 + u_{,11}^2\right) dV,\\
|u|_1^2 \equiv |u|^2_{H_2^{(1,2,2)}} &= \int_V \left(u_{,1}^2 + u_{,22}^2 + u_{,33}^2\right) dV,\\
|u|_2^2 \equiv |u|^2_{H_2^{(2,1,2)}} &= \int_V \left(u_{,2}^2 + u_{,11}^2 + u_{,33}^2\right) dV,\\
|u|_3^2 \equiv |u|^2_{H_2^{(2,2,1)}} &= \int_V \left(u_{,3}^2 + u_{,22}^2 + u_{,11}^2\right) dV.
\end{aligned}$$

Obviously, with these definitions we have

$$N(\mathbf{u}) = |u_1|_1^2 + |u_2|_2^2 + |u_3|_3^2. \tag{17}$$

In other words, $N(\mathbf{u})$ is a sum of squared seminorms in the corresponding Sobolev's spaces.

The total energy functional is given by

$$E(\mathbf{u}) = \frac{1}{2}N(\mathbf{u}) - A(\mathbf{u}), \quad A(\mathbf{u}) = \int\limits_V \mathbf{f}\cdot\mathbf{u}\,dV + \int\limits_{S_1} \mathbf{t}\cdot\mathbf{u}\,dS. \tag{18}$$

In what follows we introduce a weak solution as a minimizer of $E(\mathbf{u})$ on the given energy space H. First, let us consider Dirichlet-type boundary conditions.

4.1. Dirichlet-type Boundary Conditions.

Let us consider the case when S is clamped

$$\mathbf{u}\big|_S = \mathbf{0}. \tag{19}$$

Boundary condition (19) prevents the appearance of null-energy solutions (10)–(12). Thus, $N(\mathbf{u})$ is a squared norm. We introduce the energy space H_0 as follows

Definition 1. *The space H_0 is the completion with respect to the norm*

$$\|\mathbf{u}\|_H^2 = N(\mathbf{u})$$

of a set of vector functions $\mathbf{u}$ *that are continuously differentiable on V and satisfy* (19).

There exists the inner product in H_0

$$(\mathbf{u},\mathbf{v})_H = \int\limits_V \big(u_{1,1}v_{1,1} + u_{2,2}v_{2,2} + u_{3,3}v_{3,3} + u_{2,11}v_{2,11} + u_{3,11}v_{3,11} + u_{1,22}v_{1,22} + u_{3,22}v_{3,22} + u_{1,33}v_{1,33} + u_{2,33}v_{2,33}\big)\, dV. \tag{20}$$

So H_0 is a Hilbert space. Moreover, here

$$H_0 = \overset{\circ}{H}{}_2^{(1,2,2)}(V) \oplus \overset{\circ}{H}{}_2^{(2,1,2)}(V) \oplus \overset{\circ}{H}{}_2^{(2,2,1)}(V),$$

where $\overset{\circ}{H}{}_2^{\ell}(V)$ is the completion with the norm $\|\cdot\|_{H_2^\ell(V)}$ od a set of functions that satisfy (19), and ℓ is a multi-index, $\ell = (1,2,2)$, $(2,1,2)$, $(2,2,1)$.

Now we introduce a weak (generalized) solution.

Definition 2. *The element $\mathbf{u} \in H_0$ is a weak of the equilibrium problem for the body B with external loads* $\mathbf{f}$ *if it minimizes $E(\mathbf{u})$ on the energy space H_0.*

Calculating the first variation of $E(\mathbf{u})$ we conclude that a weak solution satisfies the equation

$$(\mathbf{u},\mathbf{v})_H - \int\limits_V \mathbf{f}\cdot\mathbf{v}\,dV = 0, \quad \forall\ \mathbf{v} \in H_0. \tag{21}$$

For simplicity we assume that $\mathbf{f} \in L_2(V)$. Then we have

Theorem 1. *A weak solution $\mathbf{u} \in H_0$ to equation* (23) *exists and unique.*

Proof. The proof mimics one given in [16, 17] and we omit it for brevity. □

4.2. Mixed Boundary Conditions

Let us now relax the kinematic constraints assuming that only a part $S_2 \subset S$ is clamped while on the rest $S_1 = S \backslash S_2$ a traction $\mathbf{t}$ is given:

$$\mathbf{u}\big|_{S_2} = \mathbf{0}. \tag{22}$$

For simplicity we assume that $\mathbf{t} \in L_2(S_2)$.

Unlike linear elasticity where (22) prevents rigid-body deformations, this is not the case in the framework of the gradient-incomplete strain-gradient elasticity, see few examples in [17]. Following [16, 17] we assume that assumed boundary conditions are non-singular. In other words we apply such kinematical constraints (22) that are sufficient for absence of null-energy deformations. In this case we define again a weak solution $\mathbf{u} \in H$ as a minimizer of the total energy functional $E(\mathbf{u})$. Here the energy space H the completion of a set of differentiable functions, with respect to the norm $\|\cdot\|_H$, that satisfy (22). H is a Hilbert space with inner product (20). Moreover, $H = H_2^{(1,2,2)}(V) \oplus H_2^{(2,1,2)}(V) \oplus H_2^{(2,2,1)}(V)$. Similarly to [14, 17] we can prove

Theorem 2. *Suppose the boundary conditions are non-singular. There exists a weak solution $\mathbf{u} \in H$ to the equilibrium problem that minimizes the total energy functional $E(\mathbf{u})$ in H and satisfies the integral equation*

$$(\mathbf{u}, \mathbf{v})_H - \int\limits_V \mathbf{f} \cdot \mathbf{v}\, dV - \int\limits_{S_1} \mathbf{t} \cdot \mathbf{v}\, dS = 0, \quad \forall\ \mathbf{v} \in H. \tag{23}$$

The weak solution is unique.

5. CONCLUSIONS

Following [16, 17] we discussed the existence and uniqueness of weak solutions in the framework of the spatial gradient-incomplete strain-gradient elasticity. To this end we used the anisotropic Sobolev spaces. In this paper we considered non-singular boundary conditions that prevent null-energy solutions. The case of singular boundary conditions and free bodies can be also studied as in [14]. Clearly, in this case a weak solution is unique up to arbitrary rigid-body deformations given by (10)–(12) whereas external loadings including hyper stresses should be orthogonal to these deformations.

Acknowledgments. Authors acknowledge the support of the Government of the Russian Federation (contract No. 14.Y26.31.0031).

REFERENCES

1. Abali BE, Müller WH, dell'Isola F (2017) Theory and computation of higher gradient elasticity theories based on action principles. Archive of Applied Mechanics 87(9):1495–1510
2. Adams RA, Fournier JJF (2003) Sobolev Spaces, Pure and Applied Mathematics, vol 140, 2nd edn. Academic Press, Amsterdam
3. Auffray N, dell'Isola F, Eremeyev VA, Madeo A, Rosi G (2015) Analytical continuum mechanics à la Hamilton–Piola least action principle for second gradient continua and capillary fluids. Mathematics and Mechanics of Solids 20(4):375–417
4. Besov OV, Il'in VP, Nikol'skii SM (1978) Integral Representations of Functions and Imbedding Theorems, vol 1. Wiley, New York
5. Besov OV, Il'in VP, Nikol'skii SM (1979) Integral Representations of Functions and Imbedding Theorems, vol 2. Wiley, New York
6. Boutin C, dell'Isola F, Giorgio I, Placidi L (2017) Linear pantographic sheets: Asymptotic micro-macro models identification. Mathematics and Mechanics of Complex Systems 5(2):127–162
7. Chandrasekhar S (1977) Liquid Crystals. Cambridge University Press, Cambridge
8. Ciarlet PG (1988) Mathematical Elasticity. Vol. I: Three-Dimensional Elasticity. North-Holland, Amsterdam
9. dell'Isola F, Andreaus U, Placidi L (2015) At the origins and in the vanguard of peridynamics, non-local and higher-gradient continuum mechanics: An underestimated and still topical contribution of Gabrio Piola. Mathematics and Mechanics of Solids 20(8):887–928

10. dell'Isola F, Della Corte A, Giorgio I (2016) Higher-gradient continua: The legacy of Piola, Mindlin, Sedov and Toupin and some future research perspectives. Mathematics and Mechanics of Solids p 1081286515616034
11. dell'Isola F, Seppecher P, Alibert JJ, Lekszycki T, Grygoruk R, Pawlikowski M, Steigmann D, Giorgio I, Andreaus U, Turco E, Gołaszewski M, Rizzi N, Boutin C, Eremeyev VA, Misra A, Placidi L, Barchiesi E, Greco L, Cuomo M, Cazzani A, Corte AD, Battista A, Scerrato D, Eremeeva IZ, Rahali Y, Ganghoffer JF, Müller W, Ganzosch G, Spagnuolo M, Pfaff A, Barcz K, Hoschke K, Neggers J, Hild F (2019) Pantographic metamaterials: an example of mathematically driven design and of its technological challenges. Continuum mechanics and Thermodynamics 31(4):851–884
12. dell'Isola F, Seppecher P, Spagnuolo M, Barchiesi E, Hild F, Lekszycki T, Giorgio I, Placidi L, Andreaus U, Cuomo M, Eugster SR, Pfaff A, Hoschke K, Langkemper R, Turco E, Sarikaya R, Misra A, De Angelo M, D'Annibale F, Bouterf A, Pinelli X, Misra A, Desmorat B, Pawlikowski M, Dupuy C, Scerrato D, Peyre P, Laudato M, Manzari L, Göransson P, Hesch C, Hesch S, Franciosi P, Dirrenberger J, Maurin F, Vangelatos Z, Grigoropoulos C, Melissinaki V, Farsari M, Muller W, Abali BE, Liebold C, Ganzosch G, Harrison P, Drobnicki R, Igumnov L, Alzahrani F, Hayat T (2019) Advances in pantographic structures: design, manufacturing, models, experiments and image analyses. Continuum Mechanics and Thermodynamics 31(4):1231–1282
13. Eremeyev VA, dell'Isola F (2018) A note on reduced strain gradient elasticity. In: Altenbach H, Pouget J, Rousseau M, Collet B, Michelitsch T (eds) Generalized Models and Non-classical Approaches in Complex Materials 1, Advanced Structured Materials, vol 89, Springer, Cham, pp 301–310
14. Eremeyev VA, Lebedev LP (2013) Existence of weak solutions in elasticity. Mathematics and Mechanics of Solids 18(2):204–217
15. Eremeyev VA, Cloud MJ, Lebedev LP (2018) Applications of Tensor Analysis in Continuum Mechanics. World Scientific, New Jersey
16. Eremeyev VA, dell'Isola F, Boutin C, Steigmann D (2018) Linear pantographic sheets: existence and uniqueness of weak solutions. Journal of Elasticity 132(2):175–196
17. Eremeyev VA, Alzahrani FS, Cazzani A, dell'Isola F, Hayat T, Turco E, Konopińska-Zmysłowska V (2019) On existence and uniqueness of weak solutions for linear pantographic beam lattices models. Continuum Mechanics and Thermodynamics 31(6):1843–1861
18. Fichera G (1972) Existence theorems in elasticity. In: Flügge S (ed) Handbuch der Physik, vol VIa/2, Springer, Berlin, pp 347–389
19. de Gennes PG, Prost J (1993) The Physics of Liquid Crystals, 2nd edn. Clarendon Press, Oxford
20. Healey TJ, Krömer S (2009) Injective weak solutions in second-gradient nonlinear elasticity. ESAIM: Control, Optimisation and Calculus of Variations 15(4):863–871
21. Lebedev LP, Vorovich II (2003) Functional Analysis in Mechanics. Springer, New York
22. Mareno A, Healey TJ (2006) Global continuation in second-gradient nonlinear elasticity. SIAM J Math Analysis 38(1):103–115
23. Maugin GA (2013) Generalized continuum mechanics: Various paths. In: Continuum Mechanics Through the Twentieth Century: A Concise Historical Perspective, Springer, Dordrecht, pp 223–241
24. Maugin GA (2017) Non-Classical Continuum Mechanics: A Dictionary. Springer, Singapore
25. Maz'ya V (2011) Sobolev Spaces: with Applications to Elliptic Partial Differential Equations, Grundlehren der mathematischen Wissenschaften, vol 342, 2nd edn. Springer, berlin
26. Mindlin RD (1964) Micro-structure in linear elasticity. Archive for Rational Mechanics and Analysis 16(1):51–78
27. Mindlin RD, Eshel NN (1968) On first strain-gradient theories in linear elasticity. International Journal of Solids and Structures 4(1):109–124
28. Nikol'skii SM (1961) On imbedding, continuation and approximation theorems for differentiable functions of several variables. Russian Mathematical Surveys 16(5):55
29. Oswald P, Pieranski P (2006) Smectic and Columnar Liquid Crystals. The Liquid Crystals book series, CRC Press, Boca Raton
30. Placidi L, Andreaus U, Della Corte A, Lekszycki T (2015) Gedanken experiments for the determination of two-dimensional linear second gradient elasticity coefficients. Zeitschrift für angewandte Mathematik und Physik 66(6):3699–3725
31. Placidi L, Greco L, Bucci S, Turco E, Rizzi NL (2016) A second gradient formulation for a 2D fabric sheet with inextensible fibres. Zeitschrift für angewandte Mathematik und Physik 67(5):114
32. Simmonds JG (1994) A Brief on Tensor Analysis, 2nd edn. Springer, New Yourk
33. Toupin RA (1962) Elastic materials with couple-stresses. Archive for Rational Mechanics and Analysis 11(1):385–414
34. Triebel H (2006) Theory of Function Spaces III, Monographs in Mathematics, vol 100. Birkhäuser, Basel